Instanton effects in string/M-theory from 3d superconformal field theories

Masazumi Honda

Harish-Chandra Research Institute

References: **[M.H.-Okuyama, 1405.3653] [M.H.-Moriyama, 1404.0676]**
[Hatsuda-M.H.-Moriyama-Okuyama, 1306.4297]

+ recent papers by

Calvo, Codesido, Grassi, Hatsuda, Kallen, Marino, Matsumoto, Moriyama, Nosaka, Okuyama, Putrov, Yamazaki and Zakany

19th, Dec. Indian Strings Meeting 2014

Non-perturbative effects in string/M-theory

$\supset$Worldsheet, D-brane and membrane instantons

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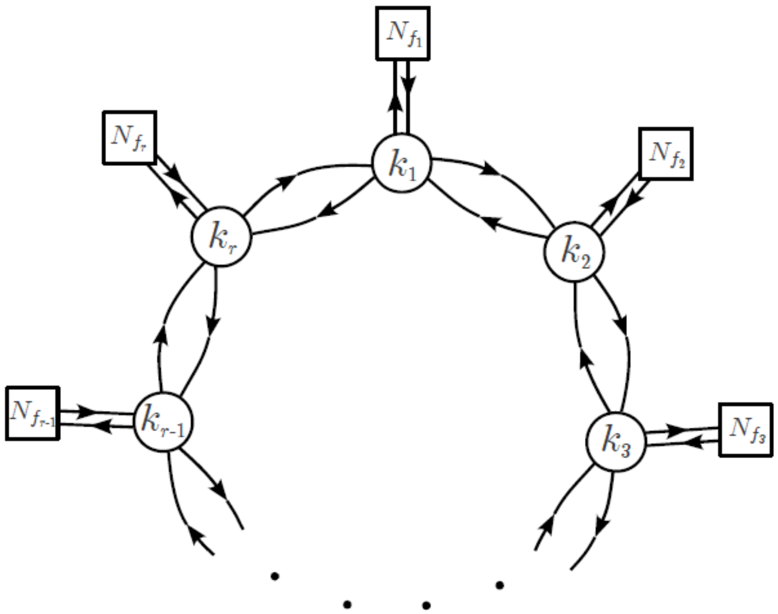
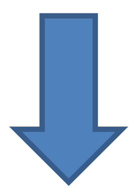
In this talk, I will report

low-energy effective theories of M2-branes provide good laboratory to probe these effects via AdS/CFT.

[The figure is borrowed from Marino-Putrov]



M2-branes w/ fractional M2-branes in certain space



3d necklace quiver Chern-Simons matter theory

[Aharony-Bergman-Jafferis-Maldacena '08,
Aharony-Bergman-Jafferis '08]

(N M2-branes) + (M fractional M2-branes) on $\mathbf{R^8/Z_k}$

(=M5-branes wrapped on $S^3/Z_k \subset R^8/Z_k$)

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Effective theory = ABJ(M) theory:

3d $\mathcal{N}=6$ $U(N)_k \times U(N+M)_{-k}$ superconformal Chern-Simons theory

(k: CS level)

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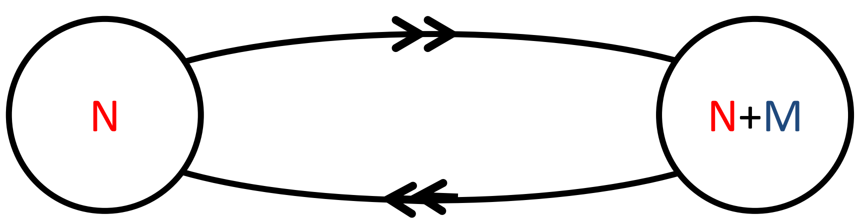
(k: CS level)

$\supset$

- Vector multiplet
- 2 bi-fundamental chiral multiplets
- 2 anti-bi-fundamental chiral multiplets

(in 3d $\mathcal{N}=2$ language)



CFT_3 / AdS_4

$U(N)_k \times U(N+M)_{-k}$

ABJ theory

CFT_3 / AdS_4

M-theory

on $AdS_4 \times S^7/Z_k$

with $\frac{1}{2\pi}\int_{S^3/Z_k} C_3 = \frac{1}{2} - \frac{M}{k}$

$k \ll N^{1/5}$

$U(N)_k \times U(N+M)_{-k}$

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ABJ theory

$\lambda = \frac{N}{k} = \text{fixed}, \ N \gg 1$

Type IIA superstring

on $AdS_4 \times CP^3$

with $\frac{1}{2\pi}\int_{CP^1} B_2 = \frac{1}{2} - \frac{M}{k}$

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with $\frac{1}{2\pi}\int_{CP^1} B_2 = \frac{1}{2} - \frac{M}{k}$

Another limit I don't consider here: $\frac{M}{k} = \text{fixed}, M \gg 1$

⟺ $\mathcal{N} = 6$ Vasiliev theory on AdS_4

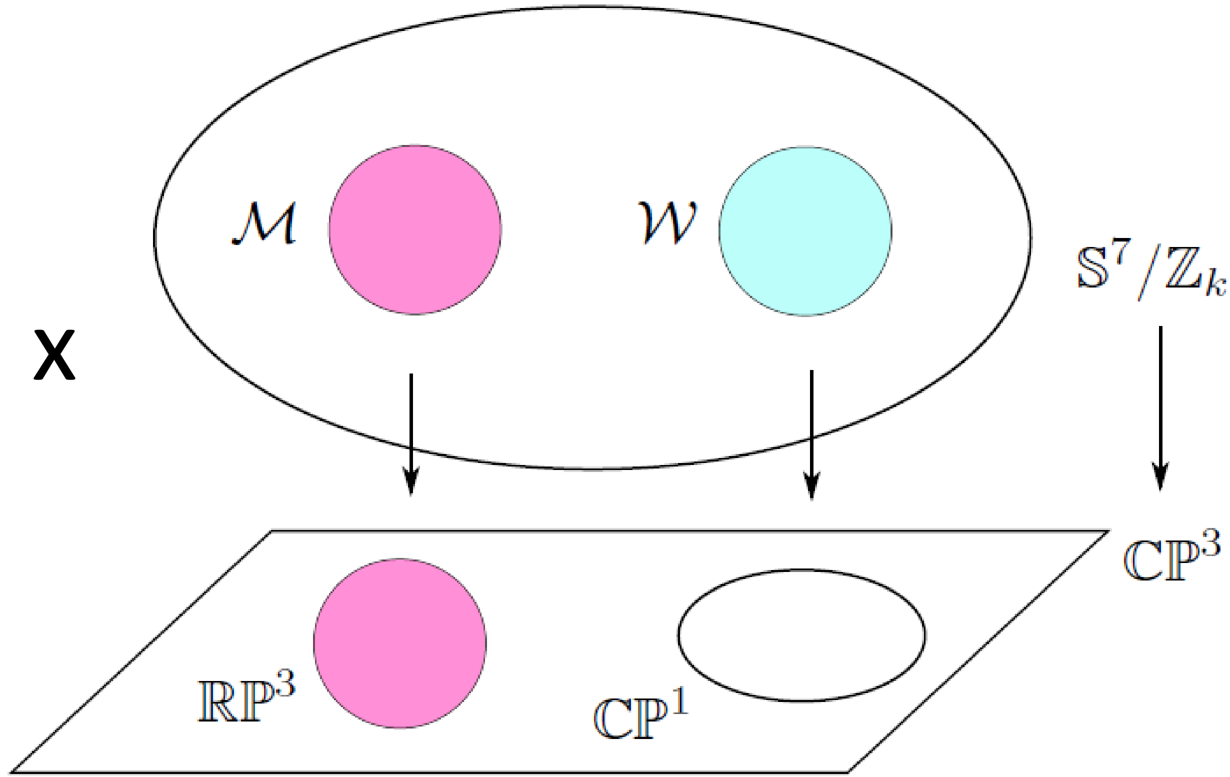
[Giombi-Minwalla-Prakash-Trivedi-Wadia-Yin '11,Chang-Minwalla-Sharma-Yin '12]

[Hirano-M.H.-Okuyama-Shigemori, to appear]

[The figure is borrowed from Hatsuda-Marino-Moriyama-Okuyama]

[cf. Cagnazzo-Sorokin-Wulff '09, Drukker-Marino-Putrov '11]

AdS_4 x



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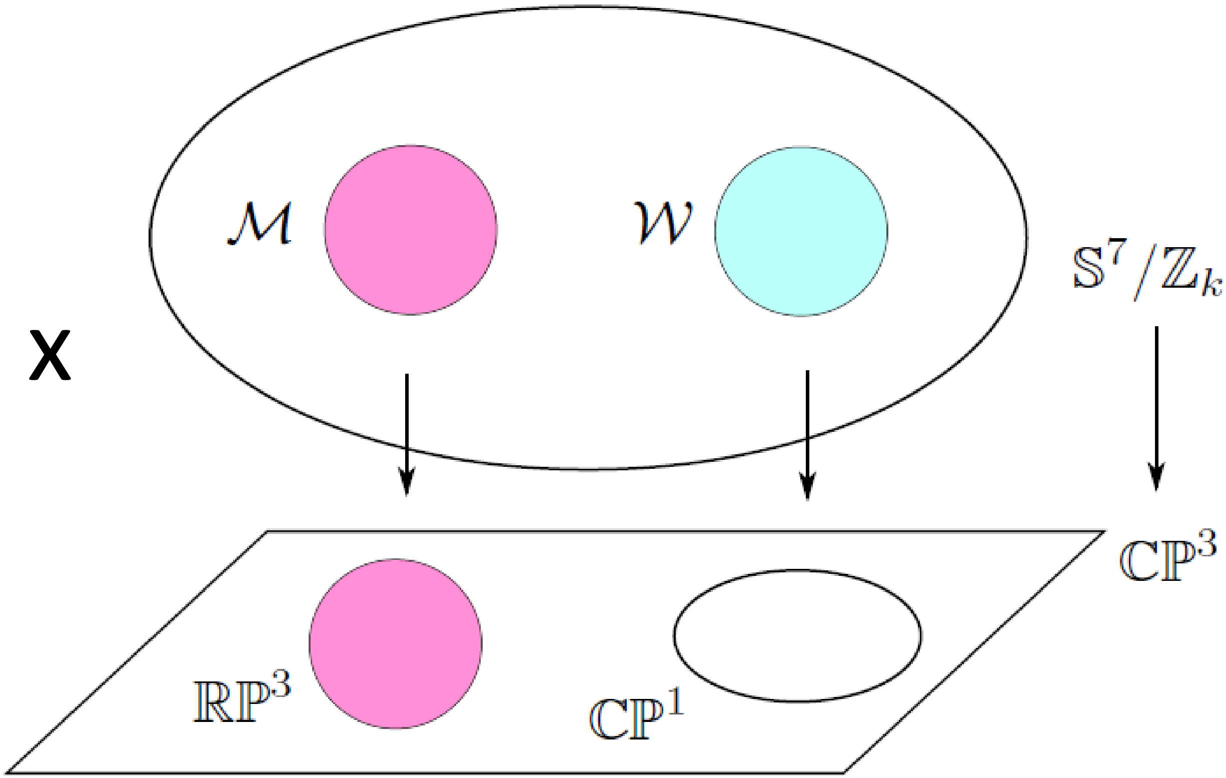

D2-brane instanton:

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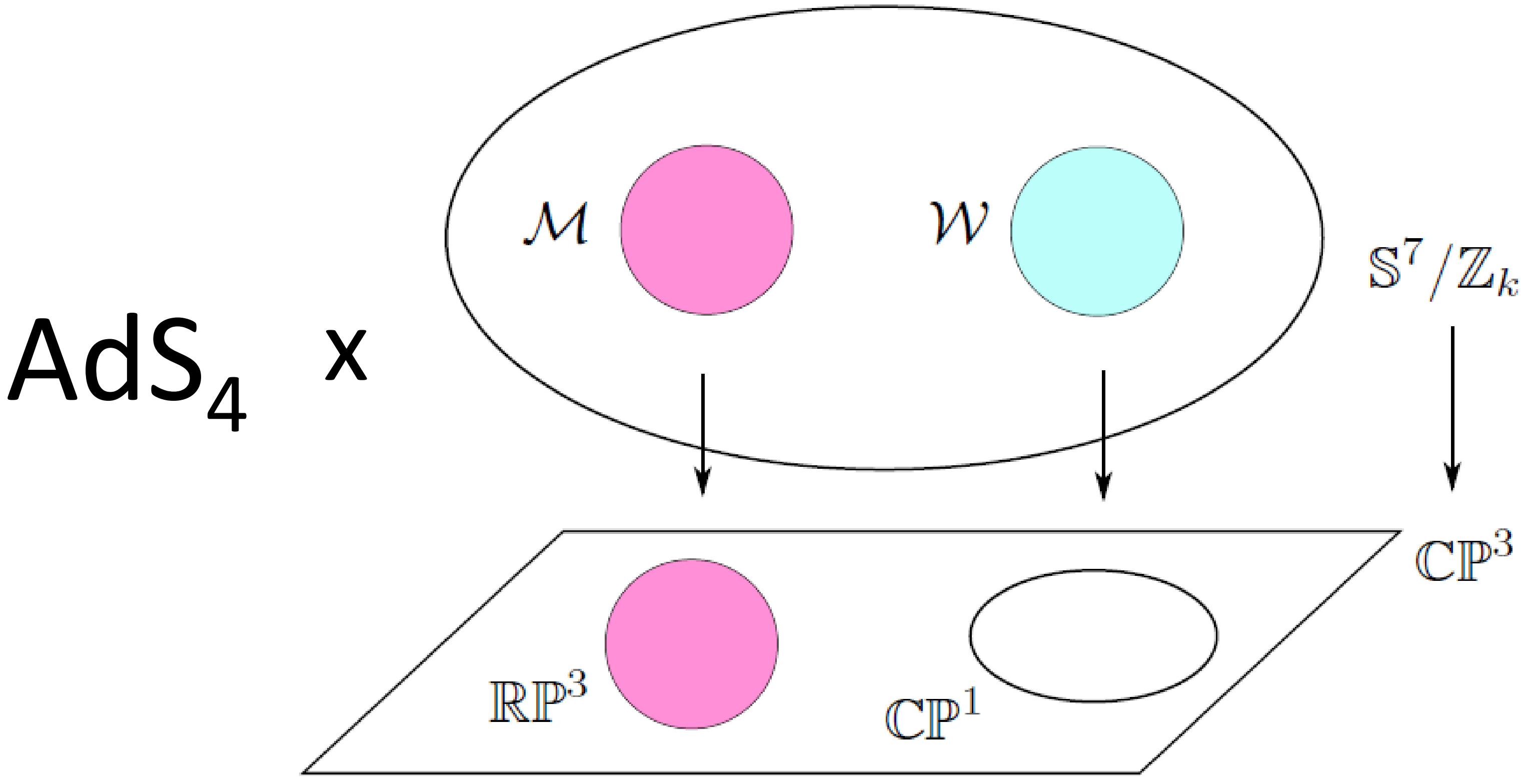

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Worldsheet instanton:

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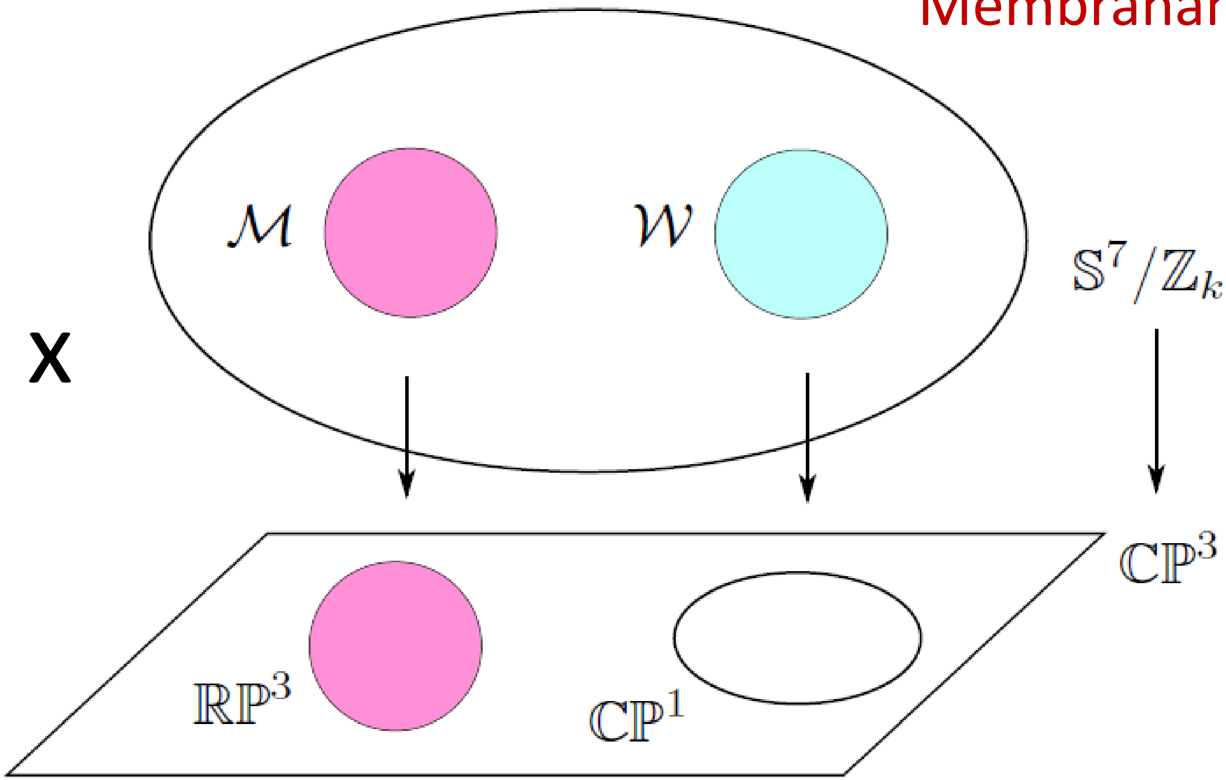

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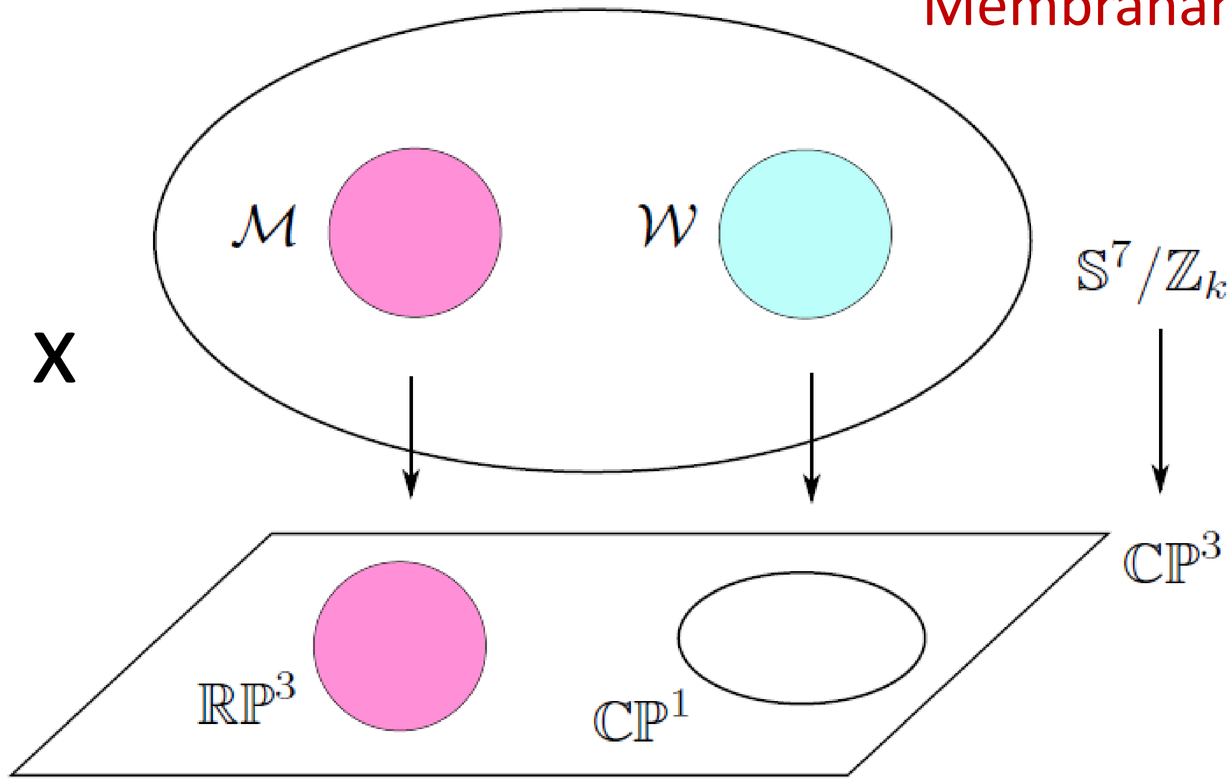

D2-brane instanton:

$$\exp\left[-T_{D2}\text{Vol}(RP^3)\right] = \exp\left[-\pi\sqrt{\frac{2N^2}{\lambda}}\right]$$

Worldsheet instanton:

$$\exp\left[-\frac{1}{2\pi\alpha'}\text{Area}(CP^1)\right] = \exp\left[-2\pi\sqrt{2\lambda}\right]$$

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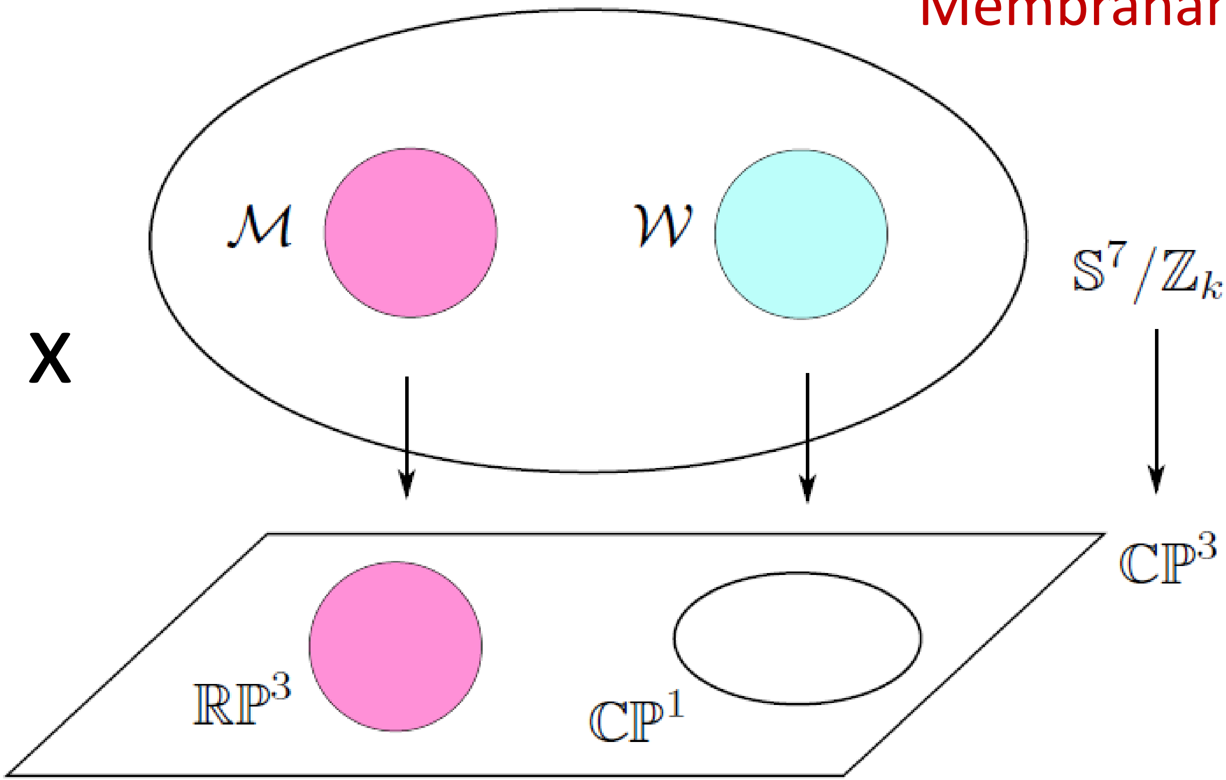

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non-perturbative in the sense of genus expansion!!

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  Ex.) For (k,M)=(2,1) up to N=65, etc…

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Some generalizations:

- BPS Wilson loop [Grassi-Kallen-Marino, Hatsuda-M.H.-Moriyama-Okuyama]

- Less SUSY theories [M.H.-Moriyama, Grassi-Marino, Hatsuda-Okuyama, Moriyama-Nosaka]

Main result

$$e^{J(\mu)} \sim \sum_N Z_{\mathrm{ABJ}}(k, M; N) e^{\mu N},$$

$$J(\mu) = J_{\mathrm{perturbative}}(\mu) + J_{\mathrm{WS-inst}}(\mu) + J_{\mathrm{D2-inst}}(\mu) + J_{\mathrm{mixed}}(\mu)$$

standard topological string

refined topological string
(in Nekrasov-Shatashvili limit)

$$Z_{\mathrm{D2},\ell-\mathrm{inst};\mathrm{WS},\mathrm{m}-\mathrm{inst}} = g_{\ell,m}\left(k, M; \frac{\partial}{\partial N}\right) \mathrm{Ai}\left[C^{-\frac{1}{3}}(k)\left(N - B(k,M) + 2\ell + \frac{4m}{k}\right)\right]$$

$$\left(\frac{Z_{\mathrm{D2},\ell-\mathrm{inst};\mathrm{WS},\mathrm{m}-\mathrm{inst}}}{Z_{\mathrm{perturbative}}} \sim e^{-\pi\ell\sqrt{2kN} - 2\pi m\sqrt{\frac{2N}{k}}}\right)$$

Instanton effects from ABJ(M) partition function

$$Z_{\mathsf{ABJ(M)}} = \int [D\Phi] \; e^{-S_{\mathsf{ABJ(M)}}[\Phi]}$$

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In this representation,

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(inconvenient to study the instantons)

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SUSY Localization

[Kapustin-Willett-Yaakov, Jafferis, Hama-Hosomichi-Lee]

$$Z_{\mathrm{ABJ(M)}} = (\mathrm{Finite\ dimentional\ integral})$$

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$$Z_{\mathrm{ABJ(M)}} = (\text{Finite dimentional integral})$$

Standard matrix model technique is available to study genus expansion,
which is convenient to study worldsheet instanton $\mathcal{O}(e^{-2\pi\sqrt{2\lambda}})$,
but not D2-instanton $\mathcal{O}(e^{-\pi\sqrt{2N^2/\lambda}})$,

ABJ(M) theory as a Fermi gas

[Marino-Putrov, Okuyama, Awata-Hirano-Shigemori, M.H.]

Localization + some explicit calculations lead us to

$$\left\{ \begin{array}{l} \hat{Z}^{(N,N+M)}(k) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int_{-\infty}^{\infty} \frac{d^N y}{(4\pi k)^N} \prod_{a=1}^{N} \rho(y_a, y_{\sigma(a)}), \\ \rho(x,y) = \frac{\sqrt{V(x)V(y)}}{\cosh \frac{x-y}{2k}}. \qquad V(x) = \frac{1}{e^{\frac{x}{2}} + (-1)^M e^{-\frac{x}{2}}} \prod_{s=-\frac{M-1}{2}}^{\frac{M-1}{2}} \tanh \frac{x + 2\pi i s}{2|k|}. \end{array} \right.$$

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Ideal Fermi gas!!

Switch to grand canonical formalism

$$\Xi_k^{(M)}(\mu) = \sum_{N=0}^{\infty} e^{\mu N}\hat{Z}^{(N,N+M)}(k) = \mathrm{Det}\left[1+e^{\mu}\rho\right]$$

ABJ(M) Fermi gas as QM

Quantum mechanical description:

$$\rho(x,y) = \langle x | e^{-\hat{H}(\hat{q},\hat{p})} | y \rangle, \quad e^{-\hat{H}(\hat{q},\hat{p})} = \sqrt{V(\hat{q})} \frac{1}{2\cosh\frac{\hat{p}}{2}} \sqrt{V(\hat{q})}, \qquad [\widehat{q}, \widehat{p}] = 2\pi i k,$$

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CS level k can be regarded as Planck constant: $\hbar = 2\pi k$

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Semi-classical expansion

||

Expansion in M-theory regime

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Expansion in M-theory regime

In this expansion,

D2-instanton: $\mathcal{O}(e^{-\pi\sqrt{2kN}})$ appears perturbatively

but not for worldsheet instanton: $\mathcal{O}(e^{-2\pi\sqrt{2N/k}})$

Simple derivation of $N^{3/2}$ law

[Marino-Putrov]

[cf. M.H.-Okuyama, Drukker-Marino-Putrov,
Herzog-Klebanov-Pufu-Teseleanu]

$$Z^{(N,N+M)}_{\mathrm{ABJ}}(k) = \int d\mu\ e^{J^{(M)}_k(\mu) - N\mu}$$

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$$Z_{\mathrm{ABJ}}^{(N,N+M)}(k) = \int d\mu \; e^{J_k^{(M)}(\mu) - N\mu}$$

$$\Downarrow \quad N \to \infty$$

$$\log \hat{Z}^{(N,N+M)}(k) \simeq J_k^{(M)}(\mu_*) - \mu_* N, \qquad \text{with} \quad \left. \frac{\partial J_k^{(M)}(\mu)}{\partial \mu} \right|_{\mu=\mu_*} = N.$$

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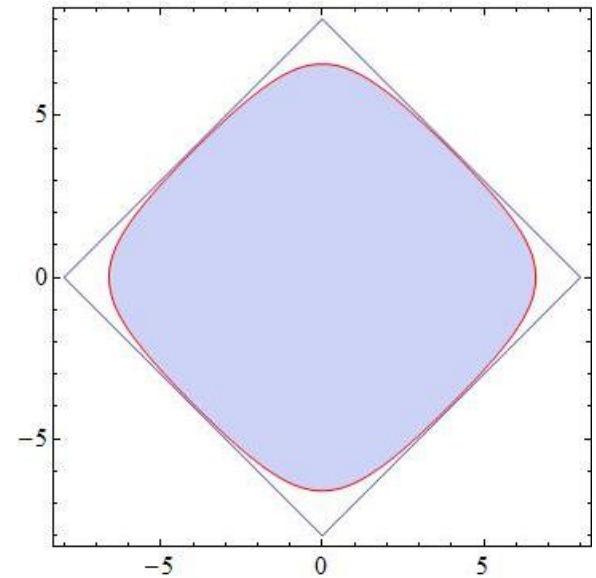
Classical Hamiltonian:

$$H_{\text{cl}}(q,p) = \log\left(2\cosh\frac{q}{2}\right) + \log\left(2\cosh\frac{p}{2}\right) \sim \frac{|q|+|p|}{2}$$



H(q,p)=E=4

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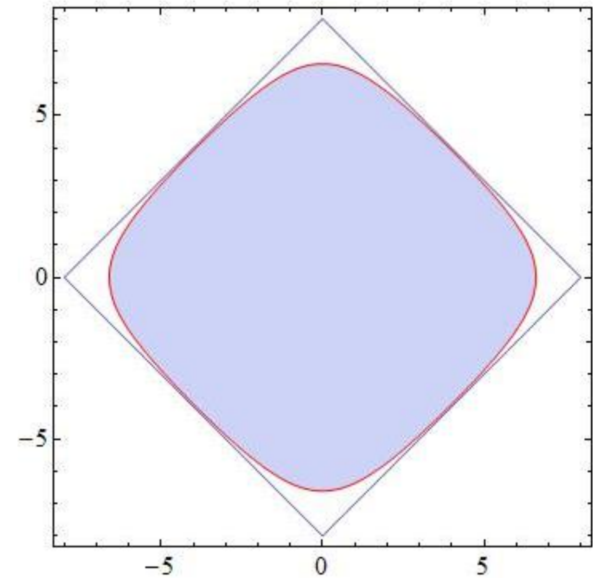
Classical grand potential:

$$J_k^{(M)}(\mu) \sim \int dE \; \frac{\text{Vol}(H_{\text{cl}} \le E)}{1+ze^{-E}} \sim \frac{2}{3\pi^2 k}\mu^3, \qquad \mu_* = \pi\sqrt{\frac{kN}{2}}$$



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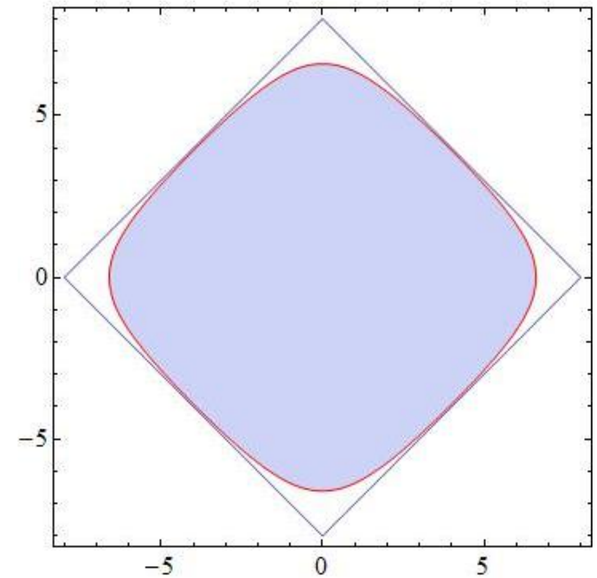

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$$\log Z_{\text{ABJ}}^{(N,N+M)}(k) \sim -\frac{\pi\sqrt{2k}}{3} N^{3/2}$$

Perturbative part

[Marino-Putrov]

[cf. Analysis by genus expansion:Fuji-Hirano-Moriyama,
Monte-Carlo: Hanada-M.H.-Honma-Nishirmura-Shiba-Yoshida]

Semi-classical analysis shows

(C,B,A: independent of μ)

$$J(\mu) = \underbrace{\frac{C}{3}\mu^3 + B\mu}_{\text{need information only on leading and sub-leading}} + \underbrace{A + (\text{instantons})}_{\text{need full order information}}$$

Perturbative part

[Marino-Putrov]

[cf. Analysis by genus expansion:Fuji-Hirano-Moriyama,
Monte-Carlo: Hanada-M.H.-Honma-Nishirmura-Shiba-Yoshida]

Semi-classical analysis shows

(C,B,A: independent of μ)

$$J(\mu) = \underbrace{\frac{C}{3}\mu^3 + B\mu}_{\text{need information only on leading and sub-leading}} + \underbrace{A + (\text{instantons})}_{\text{need full order information}}$$

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leading and sub-leading

need full order information

$$Z_{\text{pert}}(N) = \int_{-i\infty}^{i\infty} d\mu\, e^{\frac{C}{3}\mu^3 + (B-N)\mu + A} = C^{-1/3} e^{A} \text{Ai}\left[C^{-1/3}(N-B)\right]$$

This is true also for general $\mathcal{N} \geq 3$ necklace quiver.

One-loop test of AdS/CFT

$$\hat{Z}_{\rm pert}^{(N,N+M)}(k) = C^{-1/3} e^{A} {\rm Ai}[C^{-1/3}(N-B)].$$

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$N \gg 1$

$$\log \hat{Z}_{\rm pert}^{(N,N+M)}(k) = -\frac{2}{3} C^{-1/2} N^{3/2} + C^{-1/2} B N^{1/2} - \frac{1}{4}\log N + \mathcal{O}(1).$$

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classical SUGRA

universal term coming from Airy

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The logarithmic term appears in 11d SUGRA on $AdS_4 \times X_7$ at 1-loop.

[Bhattacharyya –Grassi-Marino-Sen '12]

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[Bhattacharyya –Grassi-Marino-Sen ’12]

Airy function behavior also appears from localization of the SUGRA.

[Dabholkar-Drukker-Gomes]

Exact computations

[Hatsuda-Moriyama-Okuyama,
Putrov-Yamazaki,M.H.-Okuyama]

We can also obtain exact values for various (k,M,N)
by applying integrability-like technique to the ideal Fermi gas

Ex.) For (k,M)=(2,1) up to N=65 and for (k,M)=(4,1) up to N=64, etc…

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Ex.) For (k,M)=(2,1) up to N=65 and for (k,M)=(4,1) up to N=64, etc…

Exact values for (k,M)=(2,1)

$$\hat{Z}^{(1,2)}(2)=\frac{1}{4\pi},\quad \hat{Z}^{(2,3)}(2)=\frac{1}{128}-\frac{1}{16\pi^2},\quad \hat{Z}^{(3,4)}(2)=\frac{5\pi^2-48}{4608\pi^3},$$

$$\hat{Z}^{(4,5)}(2)=\frac{9}{32768}+\frac{5}{3072\pi^4}-\frac{53}{18432\pi^2},\quad \hat{Z}^{(5,6)}(2)=\frac{6240-800\pi^2+17\pi^4}{29491200\pi^5},$$

$$\hat{Z}^{(6,7)}(2)=\frac{-218880+1413600\pi^2-1160264\pi^4+103275\pi^6}{8493465600\pi^6},$$

$$\hat{Z}^{(7,8)}(2)=\frac{-4677120-8631840\pi^2+14206864\pi^4-1345977\pi^6}{1664719257600\pi^7},$$

$$\hat{Z}^{(8,9)}(2)=\frac{61608960-1051438080\pi^2+2363612608\pi^4-1477376224\pi^6+126511875\pi^8}{213084064972800\pi^8},$$

$$\hat{Z}^{(9,10)}(2)=\frac{633830400+6140897280\pi^2-22473501120\pi^4+16465544384\pi^6-1444050207\pi^8}{23013079017062400\pi^9},$$

Exact values for (k,M)=(2,1)

[M.H.-Okuyama]

$$\hat{Z}^{(14,15)}(2)=\frac{1}{544033088978498032137928704000\pi^{14}}\Big[-555727897460736000+89573494835323699200\pi^2-1055636150467356057600\pi^4+5131488836828022789120\pi^6$$
$$-12078328057432325328640\pi^8+13537831707363614586208\pi^{10}-6051892803562043641080\pi^{12}+486239579473363340625\pi^{14}\Big],$$

$$\hat{Z}^{(15,16)}(2)=-\frac{1}{48962978008064822892413583360000\pi^{15}}\Big[36090194527715328000+610458394967156736000\pi^2-92067509353118319820800\pi^4$$
$$+507831737592928484736000\pi^6-1344043476982266371351040\pi^8+1708199914796799315018400\pi^{10}-841038818134977117865584\pi^{12}+69024176701151867566875\pi^{14}\Big],$$

$$\hat{Z}^{(16,17)}(2)=\frac{1}{1253452237006459466045787734016000000\pi^{16}}\Big[644515885825523712000-195903374317541130240000\pi^2+3317788425511538166988800\pi^4-24242575894767562235904000\pi^6$$
$$+91686579377609424295127040\pi^8-184621497276384941161625600\pi^{10}+187135567910967396538249344\pi^{12}-78705401521585216044052800\pi^{14}+6236022606745884843515625\pi^{16}\Big],$$

$$\hat{Z}^{(17,18)}(2)=\frac{1}{14489907859794671427489306205224960000000\pi^{17}}\Big[50222901705188179968000+17114872531857226334208000\pi^2-355825180591455246748876800\pi^4$$
$$+2838673897897708635616051200\pi^6-11715836542518334641324349440\pi^8+26594007390595358524134338560\pi^{10}$$
$$-31088486157208526910587238784\pi^{12}+14680941405810341458359816576\pi^{14}-1194793767361309903416444375\pi^{16}\Big],$$

$$\hat{Z}^{(18,19)}(2)=\frac{1}{1251928039086259611335076056131436544000000\pi^{18}}\Big[-2837855912505174392832000+1565466573304371781435392000\pi^2-36201047925887447842868428800\pi^4$$
$$+374743421357886210747698380800\pi^6-2099946681695866974987064688640\pi^8+6688454172020401470415744112640\pi^{10}-11982818897222541532284369726464\pi^{12}$$
$$+11220643955054903542467568447104\pi^{14}-4489098718626188671320477135000\pi^{16}+351431054003164340356323046875\pi^{18}\Big],$$

$$\hat{Z}^{(19,20)}(2)=-\frac{1}{180778408844055887876784982505379436953600000\pi^{19}}\Big[260034050935690604052480000+167378576740920004904091648000\pi^2-4603213941146778919710228480000\pi^4$$
$$+49864936569429230001889571635200\pi^6-292626274613554624545116349235200\pi^8+1022337025900122231369611246684160\pi^{10}-2112649945836780855818878981703680\pi^{12}$$
$$+2341691134873926453650025102600576\pi^{14}-1075830030189292612090801154991984\pi^{16}+87057436298005995587368943405625\pi^{18}\Big],$$

$$\hat{Z}^{(20,21)}(2)=\frac{1}{2892454541504894206028559720086070991257600000000\pi^{20}}\Big[256900007070011711972966400000-248855172342016824747491328000000\pi^2+755668170954645465216072941568000\pi^4$$
$$-106216021743324263806135055155200000\pi^6+83475984203142035463930152647065600\pi^8-388930780899716024500536537133056000\pi^{10}+1087554872133572209767569467463813120\pi^{12}$$
$$-1775932910388692220449035532375705600\pi^{14}+1558438899830276774076529628858407584\pi^{16}-597787290215170549303861405923030000\pi^{18}+46306312830726949307050906271484375\pi^{20}\Big].$$

Exact values for (k,M)=(3,1)

[M.H.-Okuyama]

$$\hat{Z}^{(1,2)}(3)=\frac{1}{12}(2\sqrt{3}-3),\quad \hat{Z}^{(2,3)}(3)=\frac{1}{432}\left(-27+14\sqrt{3}+\frac{9}{\pi}\right),\quad \hat{Z}^{(3,4)}(3)=-\frac{45+18\sqrt{3}-14\sqrt{3}\pi}{1728\pi},\quad \hat{Z}^{(4,5)}(3)=\frac{702+84(27+2\sqrt{3})\pi+(1152\sqrt{3}-2881)\pi^2}{248832\pi^2},$$

$$\hat{Z}^{(5,6)}(3)=\frac{54(14\sqrt{3}-37)+840\sqrt{3}\pi+(5797-3574\sqrt{3})\pi^2}{995328\pi^2},\quad \hat{Z}^{(6,7)}(3)=\frac{17982+162(182\sqrt{3}-1647)\pi+27(2304\sqrt{3}-5905)\pi^2-7(42110\sqrt{3}-78327)\pi^3}{322486272\pi^3},$$

$$\hat{Z}^{(7,8)}(3)=\frac{-2430(61+18\sqrt{3})+83916\sqrt{3}\pi+27(28553+16770\sqrt{3})\pi^2+(78732-335594\sqrt{3})\pi^3}{1289945088\pi^3},$$

$$\hat{Z}^{(8,9)}(3)=\frac{1472580+9072(2295+74\sqrt{3})\pi+324(91008\sqrt{3}-105709)\pi^2-168(793233+60254\sqrt{3})\pi^3+(178071703-76499136\sqrt{3})\pi^4}{371504185344\pi^4},$$

$$\hat{Z}^{(9,10)}(3)=\frac{2916(774\sqrt{3}-2857)+5533920\sqrt{3}\pi-324(152814\sqrt{3}-521209)\pi^2-48(1483097\sqrt{3}-511758)\pi^3+(226863738\sqrt{3}-370195279)\pi^4}{1486016741376\pi^4},$$

$$\hat{Z}^{(10,11)}(3)=\frac{299312820+72900(7070\sqrt{3}-171747)\pi+24300(382464\sqrt{3}-411157)\pi^2-2700(6698762\sqrt{3}-62488989)\pi^3-9(5065099200\sqrt{3}-9212744479)\pi^4+25(6475592722\sqrt{3}-11826421389)\pi^5}{4012245201715200\pi^5},$$

$$\hat{Z}^{(11,12)}(3)=\frac{-131220(28925+6642\sqrt{3})+2915854200\sqrt{3}\pi+24300(2404421+1236762\sqrt{3})\pi^2-5400(8730505\sqrt{3}-7125246)\pi^3-9(40242657395+27610808958\sqrt{3})\pi^4+50(3671699105\sqrt{3}-1298211948)\pi^5}{16048980806860800\pi^5},$$

$$\hat{Z}^{(12,13)}(3)=\frac{1}{69331597085638656 00\pi^6}\Big[25651672920+3674160(281907+4562\sqrt{3})\pi+218700(11516544\sqrt{3}-6754681)\pi^2-226800(106545861+3385430\sqrt{3})\pi^3-162(232246756800\sqrt{3}-175868541043)\pi^4$$

$$+36(3302763448131+214002197506\sqrt{3})\pi^5+25(2873091390912\sqrt{3}-6479908382207)\pi^6\Big],$$

$$\hat{Z}^{(13,14)}(3)=\frac{1}{277326388342554624 00\pi^6}\Big[787320(66366\sqrt{3}-316045)+212550156000\sqrt{3}\pi-218700(12640806\sqrt{3}-69160621)\pi^2-64800(147495509\sqrt{3}-104910390)\pi^3$$

$$+162(296550133938\sqrt{3}-1076243018035)\pi^4+360(205884495833\sqrt{3}-99859338540)\pi^5-175(1189802574054\sqrt{3}-1946635606421)\pi^6\Big],$$

$$\hat{Z}^{(14,15)}(3)=\frac{1}{14676112471087990702 0800\pi^7}\Big[9762153103080+38578680(456134\sqrt{3}-24118263)\pi+19289340(70712064\sqrt{3}-37224121)\pi^2-3572100(371449442\sqrt{3}-9412651737)\pi^3$$

$$-23814(1164740241600\sqrt{3}-760466097211)\pi^4+7938(4920941754202\sqrt{3}-40985286126129)\pi^5+9(10577414304217152\sqrt{3}-16694829965655623)\pi^6-1225(241451318806186\sqrt{3}-436239059157621)\pi^7\Big],$$

$$\hat{Z}^{(15,16)}(3)=\frac{1}{58704449884351962808 3200\pi^7}\Big[-49601160(3662825+677322\sqrt{3})+170696384888400\sqrt{3}\pi+19289340(210402593+126703602\sqrt{3})\pi^2-7144200(729601789\sqrt{3}-1620343926)\pi^3$$

$$-23814(2803076569895+2326367522214\sqrt{3})\pi^4+79380(992950495129\sqrt{3}-1601735224140)\pi^5+9(65702735219040679+49510164883503726\sqrt{3})\pi^6-2450(134862880599065\sqrt{3}-58222941612138)\pi^7\Big],$$

Exact values for (k,M)=(4,1)

[M.H.-Okuyama]

$$\hat{Z}^{(1,2)}(4)=\frac{\pi-2}{16\pi},\quad \hat{Z}^{(2,3)}(4)=\frac{12+12\pi-5\pi^2}{512\pi^2},\quad \hat{Z}^{(3,4)}(4)=\frac{-168+396\pi+202\pi^2-99\pi^3}{73728\pi^3},\quad \hat{Z}^{(4,5)}(4)=\frac{1200+4320\pi-3512\pi^2-4872\pi^3+1755\pi^4}{4718592\pi^4},$$

$$\hat{Z}^{(5,6)}(4)=\frac{-38880+241200\pi+186000\pi^2-400200\pi^3-203494\pi^4+96975\pi^5}{1887436800\pi^5},\quad \hat{Z}^{(6,7)}(4)=\frac{953280+8320320\pi-7378800\pi^2-36784800\pi^3+17373764\pi^4+27667476\pi^5-9333225\pi^6}{543581798400\pi^6},$$

$$\hat{Z}^{(7,8)}(4)=\frac{-52536960+691346880\pi+566479200\pi^2-2914304400\pi^3-2014346488\pi^4+3962357364\pi^5+2156964930\pi^6-995722875\pi^7}{426168129945600\pi^7},$$

$$\hat{Z}^{(8,9)}(4)=\frac{478759680+8468167680\pi-7157041920\pi^2-89293397760\pi^3+38961966624\pi^4+232256453184\pi^5-82822457776\pi^6-145218219408\pi^7+47021834475\pi^8}{54549520633036800\pi^8},$$

$$\hat{Z}^{(9,10)}(4)=\frac{1}{23565392913471897600\pi^9}\Big[-12959654400+320811321600\pi+249167439360\pi^2-2406136078080\pi^3-1813794333120\pi^4+7622732486880\pi^5$$
$$+5866548067808\pi^6-10329554789424\pi^7-6075970569810\pi^8+2721498152625\pi^9\Big],$$

$$\hat{Z}^{(10,11)}(4)=\frac{1}{18852314330777518080000\pi^{10}}\Big[646656998400+20855511936000\pi-15908447520000\pi^2-423480742272000\pi^3+155455887162240\pi^4+2407085602588800\pi^5$$
$$-690712514324000\pi^6-4858102787889600\pi^7+1434686348402316\pi^8+2720310664056300\pi^9-855380089265625\pi^{10}\Big],$$

$$\hat{Z}^{(11,12)}(4)=\frac{1}{36498080544385275002880000\pi^{11}}\Big[-71248933324800+3066836963097600\pi+2147272497216000\pi^2-32994976801248000\pi^3-26307684678401280\pi^4+169824342336485760\pi^5$$
$$+173665340769940800\pi^6-543644538181826400\pi^7-506552450721933352\pi^8+791312771801094444\pi^9+502106969790796050\pi^{10}-218816278991454375\pi^{11}\Big],$$

$$\hat{Z}^{(12,13)}(4)=\frac{1}{21022894393565918401658880000\pi^{12}}\Big[2305385523302400+126291787634073600\pi-846667607385600000\pi^2-4394402461709568000\pi^3+12993286572791078 40\pi^4$$
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$$-168381450188362233000\pi^{11}+51759053721397378125\pi^{12}\Big],$$

$$\hat{Z}^{(13,14)}(4)=\frac{1}{5684590644020224335808561152 0000\pi^{13}}\Big[-3266960299739136 00+23110746777403084800\pi+14222217559476326400\pi^2-29659413241521423 3600\pi^3-26273574003246425 8560\pi^4$$
$$+1724522555482742695680\pi^5+3140617642113715146240\pi^6-934641469470659423 6160\pi^7-1767587028975945443094 4\pi^8+3840269234571916127467 2\pi^9+4549375668567767917089 6\pi^{10}$$
$$-6304327269971616176522 4\pi^{11}-4297687104962919234465 0\pi^{12}+1827236979240428318062 5\pi^{13}\Big],$$

Comparison with classical SUGRA

[cf. Klebanov-Tseytlin]

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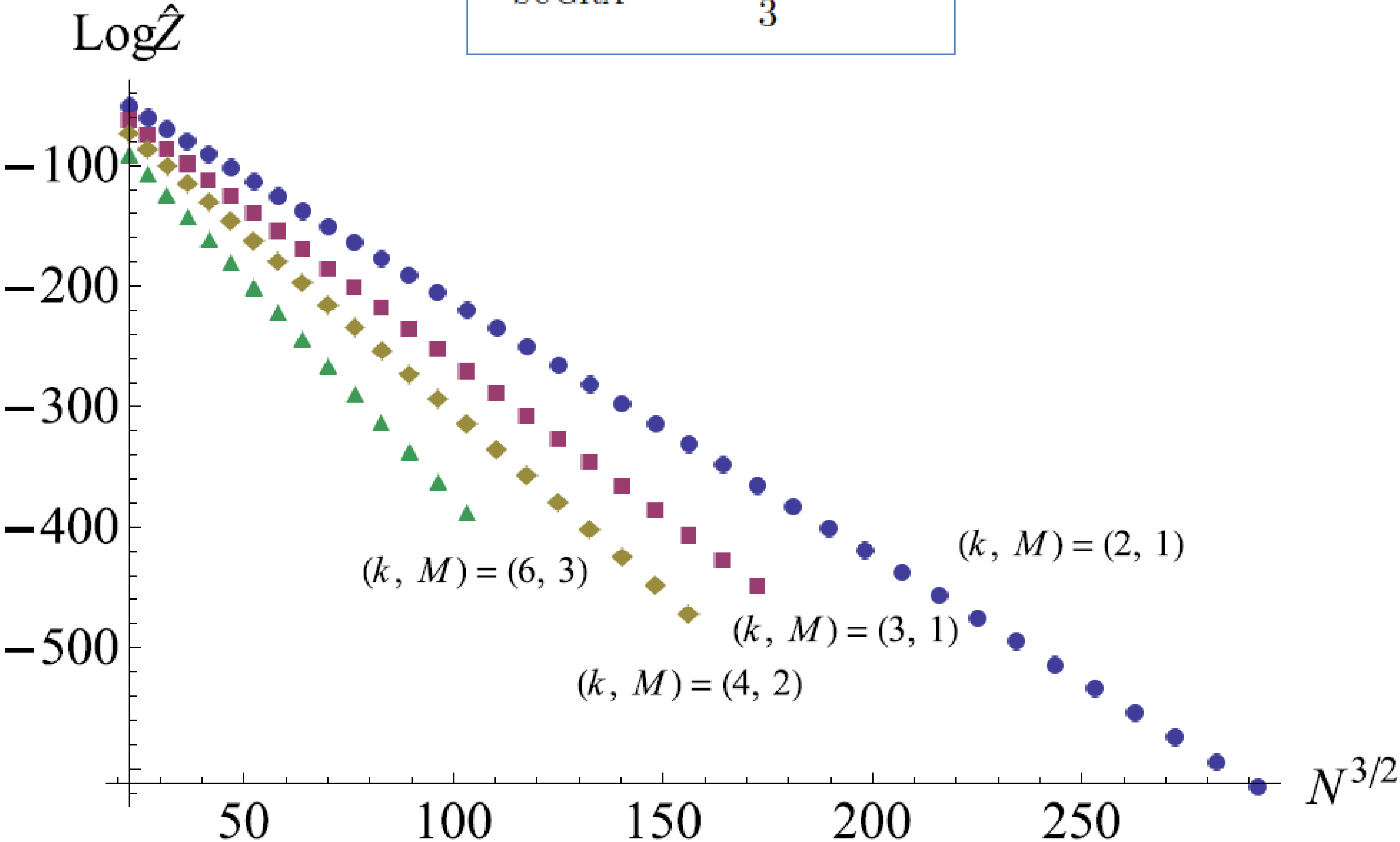

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To determine structures of non-perturbative effects completely,

we will “guess” the form of the grand potential and test this “guess” by using the above information.

Basic idea

[cf. Marino-Putrov]

ABJ(M) matrix model

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ABJ(M) matrix model

⇕ Analytic continuation

Pure CS theory on S^3/Z_2

(Lens space matrix model)

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⇕ Geometric transition
[cf. Gopakumar-Vafa '98]

Topological string on certain space (local P^1 x P^1)

Perturbative + Worldsheet instanton part

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This part is described by the standard topological string.

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$$Z_{\text{WS,m-inst}} = d_m(k,M)\text{Ai}\left[C^{-1/3}\left(N-B+\frac{4m}{k}\right)\right]$$

$$\frac{Z_{\text{WS,m-inst}}}{Z_{\text{pert}}} \sim e^{-2\pi m\sqrt{\frac{2N}{k}}}$$

Test of WS 1-instanton

$$Z^{(N,N+M)}_{\mathrm{WS},1-\mathrm{inst}}(k) = -2C^{-1/3}e^{A}\frac{\cos\pi\left(1-\frac{2M}{k}\right)}{\sin^2\frac{2\pi}{k}}\mathrm{Ai}\left[C^{-1/3}\left(B-N-\frac{4}{k}\right)\right]$$

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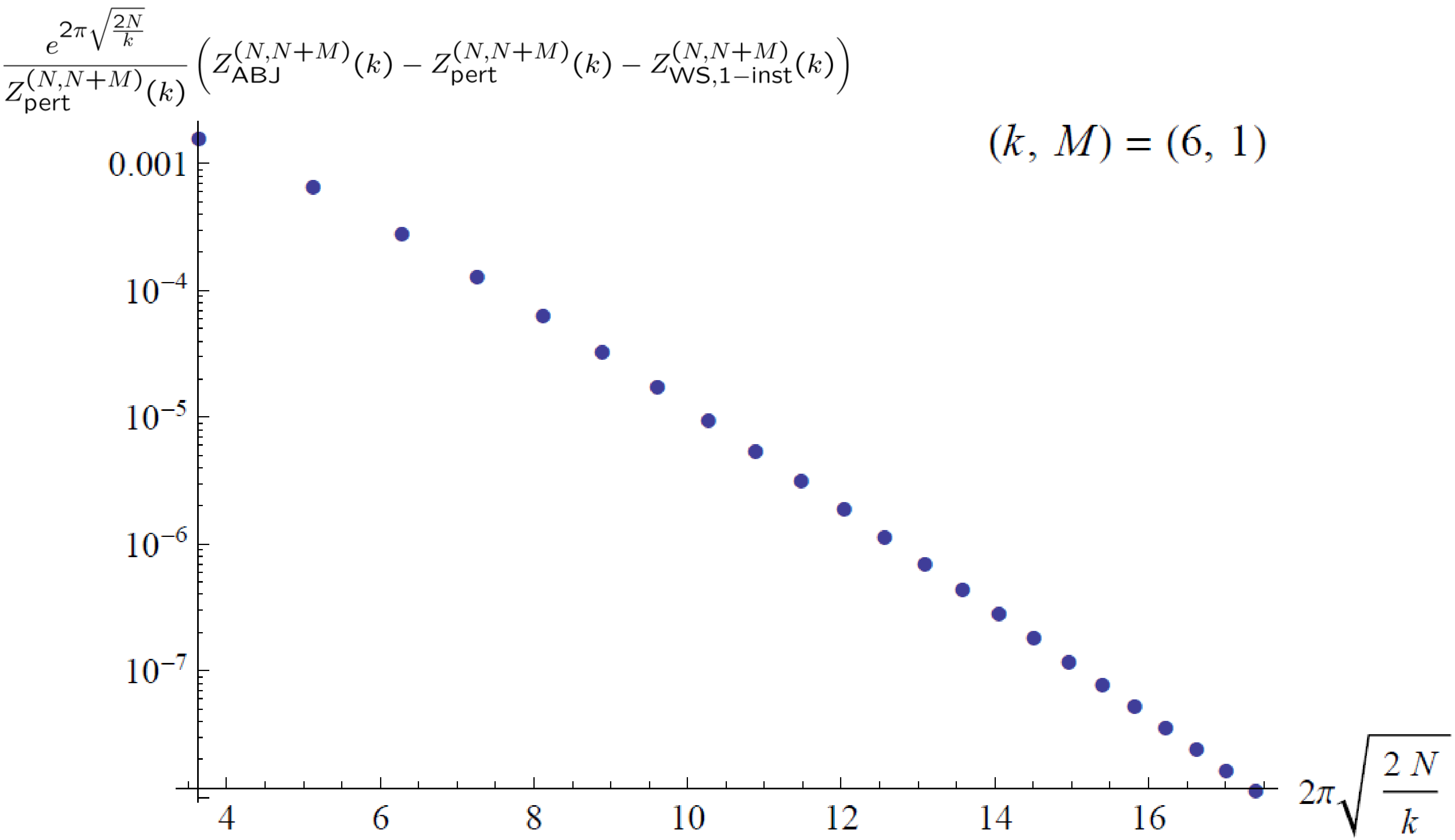

Problem on worldsheet instanton effect

[Hatsuda-Moriyama-Okuyama, Matsumoto-Moriyama, M.H.-Okuyama]

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For instance,

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This divergence must be apparent and
must cancel out if we include other sector: D2-instanton

D2-instanton + Mixture of D2- & WS-instanton part

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||

Non-perturbative in the sense of genus expansion

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Non-perturbative in the sense of genus expansion

This part is described by non-perturbative formulation of topological string: refined topological string in certain limit (Nekrasov-Shashvili limit)

[Nekrasov-Shatashvili]

D2-instanton + Mixture of D2- & WS-instanton part

| |

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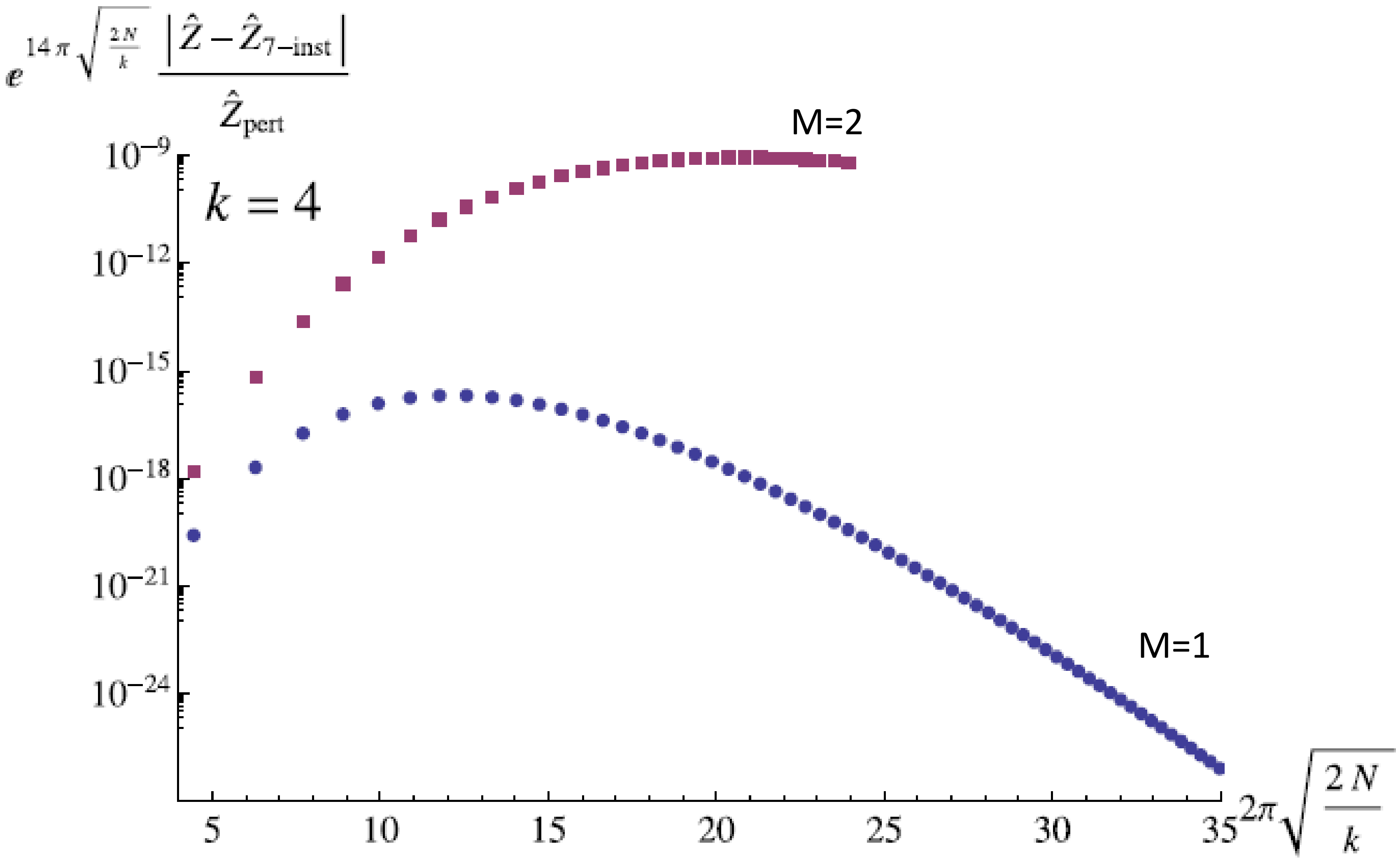
[Nekrasov-Shatashvili]

$$Z_{\mathrm{D2},\ell-\mathrm{inst;WS,m-inst}} = g_{\ell,m}\left(k, M; \frac{\partial}{\partial N}\right) \mathrm{Ai}\left[C^{-1/3}\left(N - B + 2\ell + \frac{4m}{k}\right)\right]$$

$$\frac{Z_{\mathrm{D2},\ell-\mathrm{inst;WS,m-inst}}}{Z_{\mathrm{pert}}} \sim e^{-\pi\ell\sqrt{2kN} - 2\pi m\sqrt{\frac{2N}{k}}}$$

Test of our proposal



Drastic simplification for $\mathcal{N}=8$ SUSY cases

[Codesido-Grassi-Marino]

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$$\begin{aligned}\Xi(\mu)|_{(k,M)=(1,0)} &= \left(\vartheta_2(\bar{\xi}/4,\bar{\tau}/4)+i\vartheta_1(\bar{\xi}/4,\bar{\tau}/4)\right) \qquad (\bar{\xi},\bar{\tau}: \text{determined by } F_0)\\ &\quad\times\exp\left[\frac{3\mu}{8}-\frac{3}{4}\log 2+F_1+F_1^{\rm NS}-\frac{1}{4\pi^2}\left(F_0-\lambda\partial_\lambda F_0+\frac{\lambda^2}{2}\partial_\lambda^2 F_0\right)\right]\end{aligned}$$

$$\Xi(\mu)|_{(k,M)=(2,0)} = \vartheta_3(\bar{\xi},\bar{\tau})\exp\left[\frac{\mu}{4}+F_1+F_1^{\rm NS}-\frac{1}{\pi^2}\left(F_0-\lambda\partial_\lambda F_0+\frac{\lambda^2}{2}\partial_\lambda^2 F_0\right)\right]$$

$$\Xi(\mu)|_{(k,M)=(2,1)} = \vartheta_1(\bar{\xi}+1/4,\bar{\tau})\exp\left[\frac{\log 2}{2}+F_1+F_1^{\rm NS}-\frac{1}{\pi^2}\left(F_0-\lambda\partial_\lambda F_0+\frac{\lambda^2}{2}\partial_\lambda^2 F_0\right)\right]$$

Resumming the 1/N-expansion in ABJM

[Grassi-Marino-Zakany]

[cf. Drukker-Marino-Putrov]

$$F_{\mathsf{ABJM}}\big|_{\mathsf{genus}\text{-}g} \sim (2g)!$$ asymptotic

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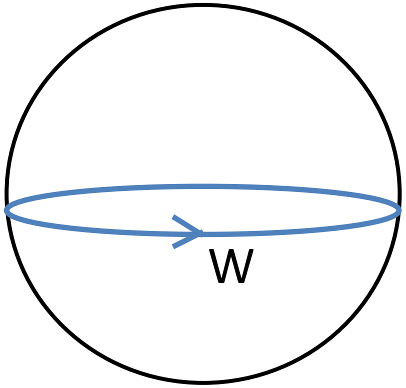
—— No, Grassi-Marino-Zakany have found relevant differences.

We should resum each string perturbation series
around each D2-instanton background (to get full result).

Some generalizations

Half-BPS Wilson loop in ABJM

[Hatsuda-M.H.-Moriyama-Okuyama, Grassi-Kallen-Marino]



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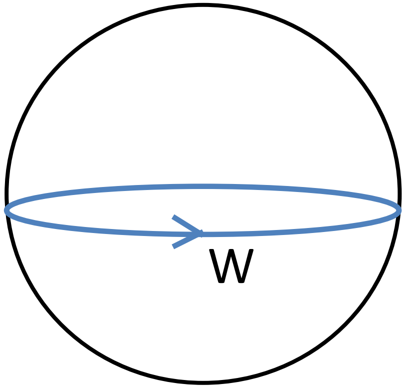
By localization + some explicit calculations,

$$\langle \text{Generating function} \rangle = (\text{Ideal Fermi gas})$$



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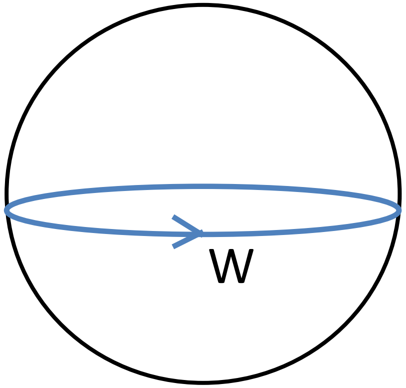

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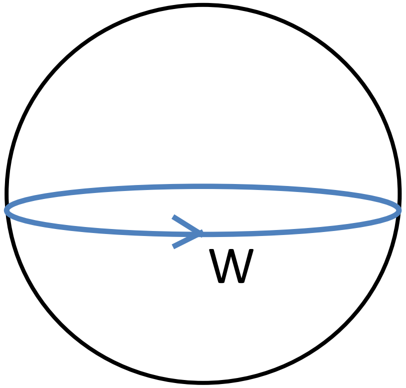

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$$Z_{\text{ABJM}} \langle W_{\mathbf{R}} \rangle_{\text{D2},\ell-\text{inst};\text{WS},\text{m}-\text{inst}} = d_{\ell,m}(k) \text{Ai} \left[C^{-\frac{1}{3}} \left(N - B + \frac{2|\mathbf{R}|}{k} + 2\ell + \frac{4m}{k} \right) \right]$$

$$\langle W_{\mathbf{R}} \rangle_{\text{D2},\ell-\text{inst};\text{WS},\text{m}-\text{inst}} \sim e^{\pi |\mathbf{R}| \sqrt{\frac{2N}{k}} - \pi \ell \sqrt{2kN} - 2\pi m \sqrt{\frac{2N}{k}}}$$

Less SUSY theories

[M.H.-Moriyama, Grassi-Marino,
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Cancelation has been found also in some $\mathcal{N}=4$ M2-brane theories (=special cases of Gaiotto-Witten theory).

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Technical difficulties for less SUSY theories:

1. Corresponding topological string is unknown.

2. Except some special cases, density matrix of Fermi gas becomes complicated (given by integral)

3. For $\mathcal{N}=2$, Fermi gas becomes interacting.

Summary & Outlook

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ABJ(M) partition function on sphere:

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Some generalizations:

- Half-BPS Wilson loop in ABJM is described by open topological string.

- Pole cancelation occurs also in some less SUSY theories.

Outlook

- ABJ theory in higher spin limit [Hirano-M.H.-Okuyama-Shigemori, to appear]

- More general M2-brane theory [Hatsuda-M.H.-Okuyama, work in progress]

- Other quantities

  Ex.) Vortex loop, Energy-momentum tensor correlator, super-Renyi entropy

- Relation to Higgs branch localization formula

  [cf. Pasquetti, Fujitsuka-M.H.-Yoshida, Benini-Peelaers]

  ――― Localization formula has another equivalent representation in terms of vortex partition functions for many 3d theories.

- Analysis on the gravity side

  ――― Test many predictions.

  Probably, localization on the gravity side and string perturbation around instanton background would be useful.

Thanks!