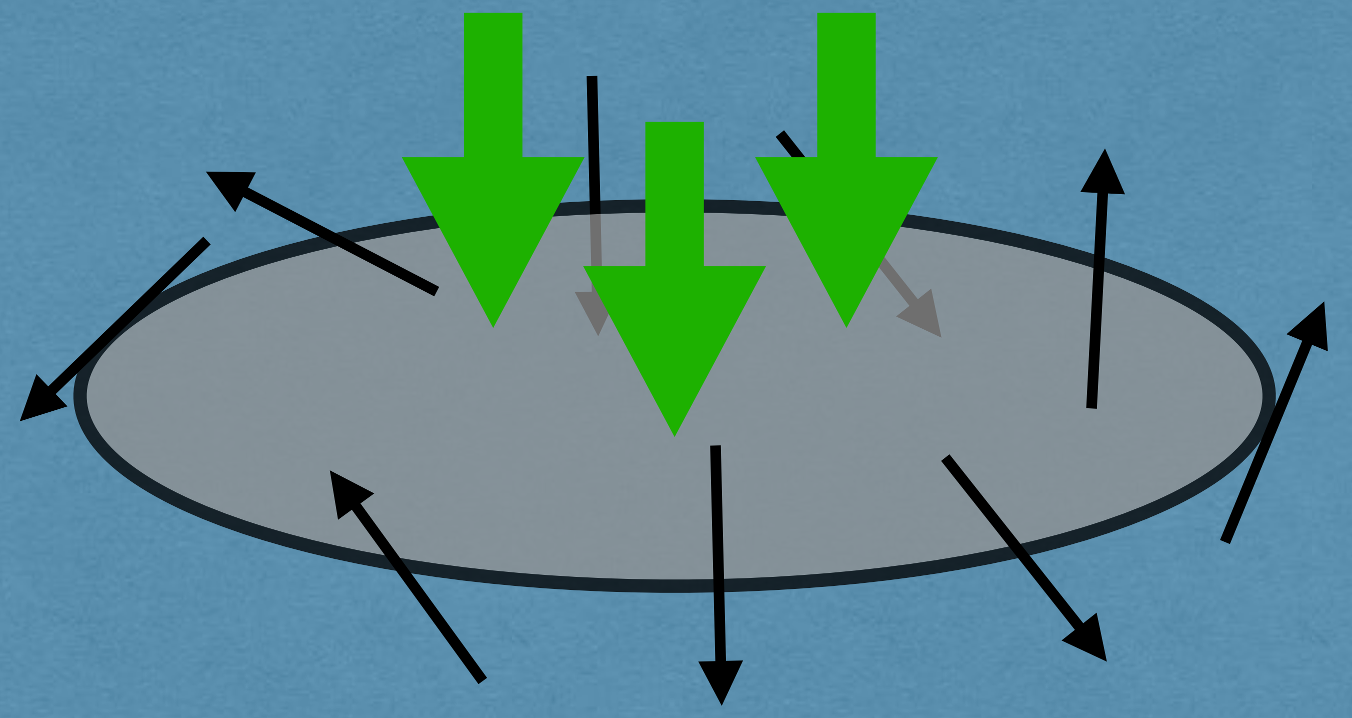
測定型量子計算と格子ゲージ理論

Measurement-based quantum computation and lattice gauge theories



場の理論の新しい計算方法2023

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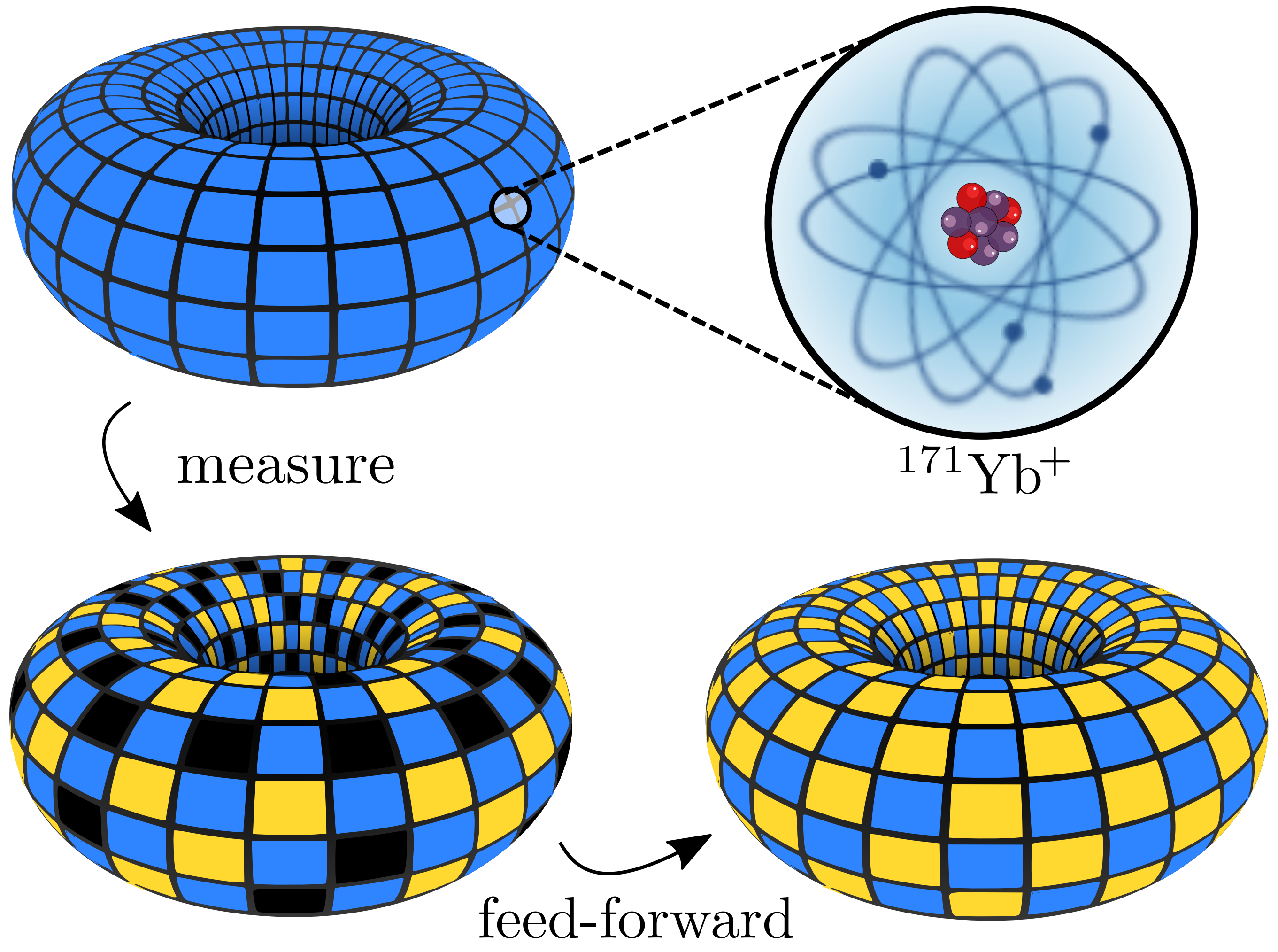
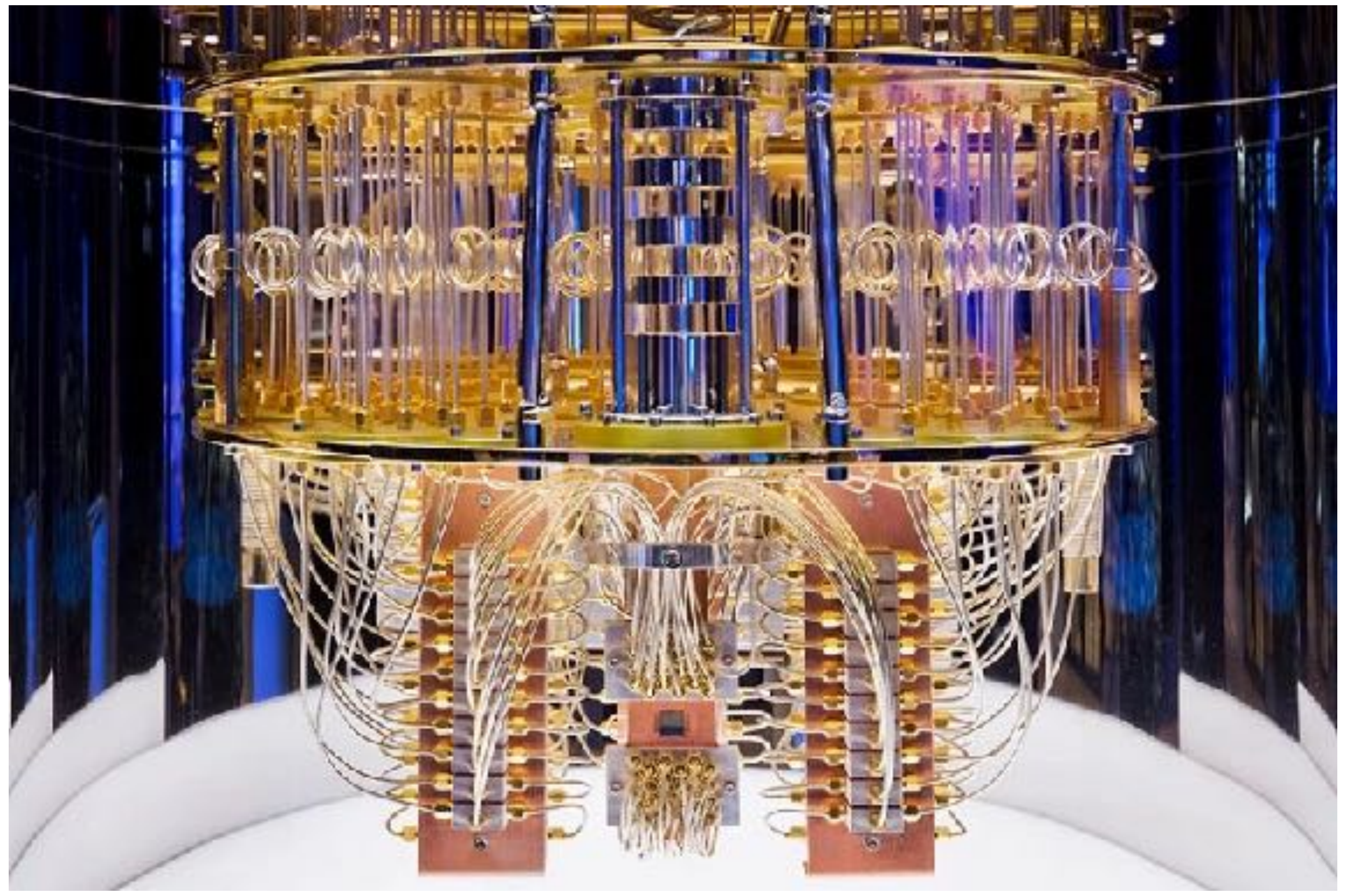
Motivation

In plethora of quantum devices, mid-circuit measurement is becoming available on cloud quantum computers.



Quantinuum
Iqbal et al. arXiv:2302.01917

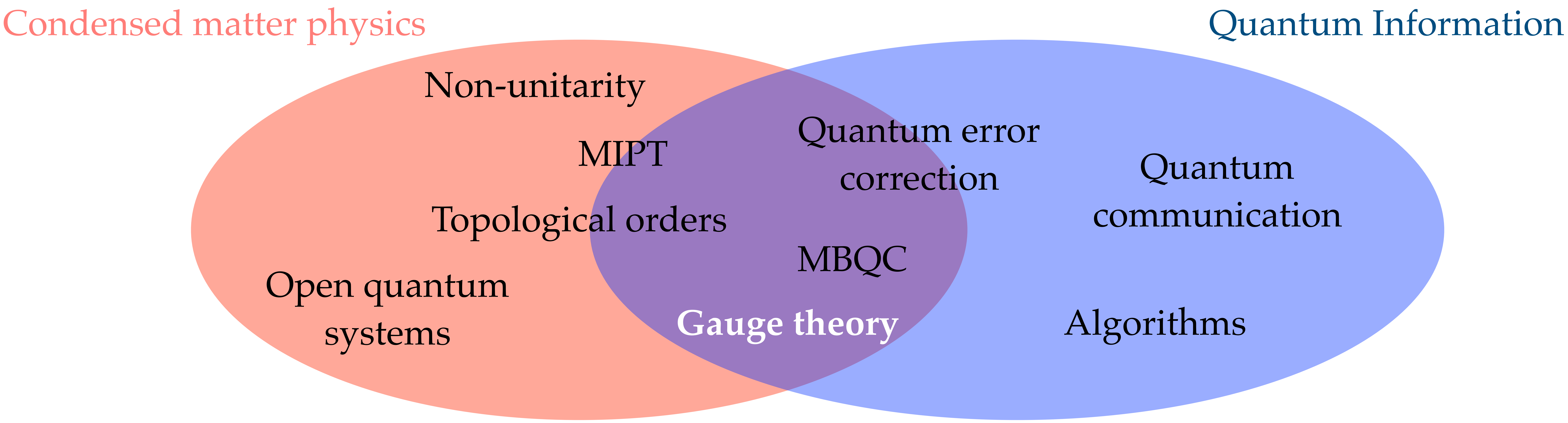


IBM Quantum
https://www.nature.com/articles/d41586-021-03476-5

QuEra

Motivation

<u>Entanglement + measurement</u>



Today's lecture aims to explain some physics and their applications woven by measurements and quantum entanglement. I will approach this topic from the perspectives of measurement-based quantum computation and lattice gauge theory.

References for beginners

Review papers/textbooks:

- 小柴, 藤井,森前『観測に基づく量子計算』コロナ社 (2017)
- M. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information,” Cambridge University Press.
- T.-C. Wei, “Quantum spin models for measurement-based quantum computation,” Advances in Physics: X, Volume 3 (2018)
- K. Fujii, “Quantum Computation with Topological Codes — from qubit to topological fault-tolerance —,” arXiv:1504.01444

Other recent papers:

- N. Tantivasadakarn, R. Thorngren, A. Vishwanath, and R. Verresen, “Long-range entanglement from measuring symmetry-protected topological phases,” arXiv:2112.01519
- H. Sukeno and T. Okuda, “Measurement-based quantum simulation of Abelian lattice gauge theories,” SciPost Physics **14** 129 (2023)

MBQC

Gate-based quantum circuit

↕

Measurement pattern on the 2d cluster state (translationally invariant graph state).

Graph state $\subset$ Stabilizer state

Plan

Part I: Quantum computation by measurement

- Stabilizer formalism
- Graph state
- Gate teleportation
- Universal quantum computation on graph states

Part II: Measurement-based quantum computation and lattice gauge theory

- Measurement as a shortcut to topological orders
- $\mathbb{Z}_2$ lattice gauge theory
- Quantum simulation of lattice gauge theories

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Stabilizer formalism

- Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\{X, Y\} = \{Y, Z\} = \{Z, X\} = 0$$

$$X^2 = Y^2 = Z^2 = I = -iXYZ$$

- Operation on Z eigenbasis

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle \quad \text{(phase-flip)}$$

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle \quad \text{(bit-flip)}$$

$$Y|0\rangle = i|1\rangle, \quad Y|1\rangle = -i|0\rangle \quad \text{(bit-flip, phase-flip, and a phase)}$$

- X eigenbasis

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \; .$$

Stabilizer formalism

- Qubit

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

- Two-qubit state

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

- n-qubit Pauli operators

$$\{\pm 1, \pm i\} \times P_1 \otimes P_2 \otimes \cdots P_n \in \mathscr{P}_n$$

$P_j \in \{I, X, Y, Z\}$.

$\mathscr{P}_n$: n-qubit Pauli group

- Example:

$$-X \otimes Z \otimes Z$$

We will also use a short hand notation such as $-X_1Z_2Z_3$.

Stabilizer formalism

- Clifford operators

Operators U that map a Pauli operator to another Pauli operator under conjugation.

$$UP_1U^\dagger = P_2 \quad (P_1, P_2 \in \mathscr{P}_n)\,.$$

- Hadamard operator H

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad HZH = X\,, \quad HXH = Z\,.$$

$$H|0\rangle = |+\rangle\,, \quad H|1\rangle = |-\rangle\,.$$

- Phase operator S

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad SXS^\dagger = Y\,.$$

Stabilizer formalism

- Controlled-NOT gate CX

$$CX_{c,t} = |0\rangle_c\langle 0|_c \otimes I_t + |1\rangle_c\langle 1|_c \otimes X_t$$

c : controlling qubit

t : target qubit

- Controlled-Z gate CZ

$$CZ_{c,t} = |0\rangle_c\langle 0|_c \otimes I_t + |1\rangle_c\langle 1|_c \otimes Z_t$$

It is a phase gate.

$|00\rangle \rightarrow |00\rangle \quad |01\rangle \rightarrow |01\rangle \quad |10\rangle \rightarrow |10\rangle \quad |11\rangle \rightarrow -|11\rangle$

Therefore, the roll of c and t is symmetric:

$$CZ_{a,b} = CZ_{b,a}$$

Stabilizer formalism

- Some algebra and mnemonic

$$CZ(I \otimes Z)CZ = I \otimes Z$$

A phase gate commutes with another phase gate.

$$CZ(I \otimes X)CZ = Z \otimes X$$

X 'triggers' the operator Z in the target qubit.

There's also a set of algebra for the CNOT gate, but I'm not going to use it today.

Stabilizer formalism

- Stabilizer group

$$\mathcal{S} = \{S_j\} \quad \text{with } S_j \in \mathcal{P} \text{ and } [S_k, S_\ell] = 0 \text{ for all elements.}$$

- Generators of a stabilizer group

The maximal set of independent stabilizers.

$$\langle \widetilde{S}_k \rangle$$

- Examples:

$$\langle IX, ZI \rangle = \{II, IX, ZI, ZX\}$$
$$\langle XX, ZZ \rangle = \{II, XX, ZZ, -YY\}$$

Stabilizer formalism

- Stabilizer state

$$S_j|\Psi\rangle = |\Psi\rangle \quad \text{for all } S_j \in \mathcal{S}.$$

- It is a simultaneous eigenstate of commuting operators.

- Examples:

$$\langle XX, ZZ\rangle \longrightarrow \text{Bell state} \quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\langle XXX, ZZI, IZZ\rangle \longrightarrow \text{GHZ state} \quad \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Graph states, which we'll define later, are also examples.

Stabilizer formalism

- A Clifford unitary or a Pauli measurement converts a stabilizer state to another stabilizer state.

- Let us start with Clifford unitaries.

Given a stabilizer state $S_j|\Psi\rangle = |\Psi\rangle$, a new stabilizer for the state $U|\Psi\rangle$ is $US_jU^\dagger$.

$$US_jU^\dagger(U|\Psi\rangle) = US_j|\Psi\rangle = U|\Psi\rangle \, .$$

Since $S_j \in \mathscr{P}$ and U is Clifford, the new stabilizer is also Pauli, $US_jU^\dagger \in \mathscr{P}$.

Measurement in stabilizer states

- Now let's look at measurement of a Pauli operator $P \in \mathcal{P}$ on stabilizer states.
- If $P \in \mathcal{S}$, then the measurement outcome is $P = +1$. The stabilizer doesn't change.
- If $P \notin \mathcal{S}$, then we reconstruct stabilizers. First, we re-group generators as

$$\mathcal{S} = \langle \underbrace{S_1, S_2, \ldots, S_k}_{\text{anti-commute with } P}, \underbrace{S_{k+1}, \ldots, S_n}_{\text{commute with } P} \rangle .$$

The measurement result of P (± 1) is random. (Probability $\frac{1}{2}$ each).

The new stabilizer is then

$$\mathcal{S}' = \langle \pm P, \underbrace{S_1 S_2, \ldots, S_1 S_k, S_{k+1}, \ldots, S_n}_{\text{commute with } P} \rangle$$

Measurement in stabilizer states

- Example 1.

$$\langle XXX, ZZI, IZZ \rangle \longrightarrow \text{GHZ state} \quad \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Measure the middle qubit in the X basis. Assume that the outcome is $X_2 = +1$.

$$\langle +X_2, X_1X_2X_3, (I_1Z_2Z_3)(Z_1Z_2I_3) \rangle$$
$$\simeq \langle +X_2, +X_1X_3, Z_1Z_3 \rangle$$
$$\longrightarrow \text{Bell } \otimes |+\rangle$$

Measurement in stabilizer states

- Example 2.

$$\langle ZXZ, XZI, IZX\rangle \longrightarrow \text{3-qubit cluster state (described later)}$$

Measure the middle qubit in the X basis. Assume that the outcome is $X_2 = +1$.

$$\begin{aligned}&\langle +X_2, Z_1X_2Z_3, (I_1Z_2X_3)(X_1Z_2I_3)\rangle\\ &\simeq \langle +X_2, +Z_1Z_3, X_1X_3\rangle\\ &\longrightarrow \text{Bell } \otimes |+\rangle\end{aligned}$$

Measurement in stabilizer states

- Example 3.

$$\langle ZXZ, XZI, IZX \rangle \quad \longrightarrow \quad \text{3-qubit graph state (described later)}$$

Measure the qubit-2 in the Z basis. Assume that the outcome is $Z_2 = +1$.

$$\langle +Z_2, I_1Z_2X_3, X_1Z_2I_3 \rangle$$
$$\simeq \langle +Z_2, X_3, X_1 \rangle$$
$$\longrightarrow |+\rangle \otimes |0\rangle \otimes |+\rangle$$

Universal quantum computation

- Gottesman-Knill theorem

Stabilizer circuits

Inputs : Pauli product basis

Circuit: Clifford gates or Pauli measurements

Stabilizer circuits can be efficiently simulated by classical computers.

- Potentially classically hard circuit:

One can decompose an arbitrary n-qubit gate to a product of universal gates.

(It could be an exponential number of gates; efficiency not guaranteed.)

- {(single qubit) SU(2) gate} ∪ {CNOT} is a *universal gate set*.
- cf. Solovay-Kitaev theorem: SU(2) can be efficiently approximated by $\{H, e^{i\pi/8}\}$ to arbitrary accuracy.

MBQC

Universal quantum computation

$\updownarrow$

Measurement on the 2d cluster state
(translationally invariant graph state).

Graph state $\subset$ Stabilizer state

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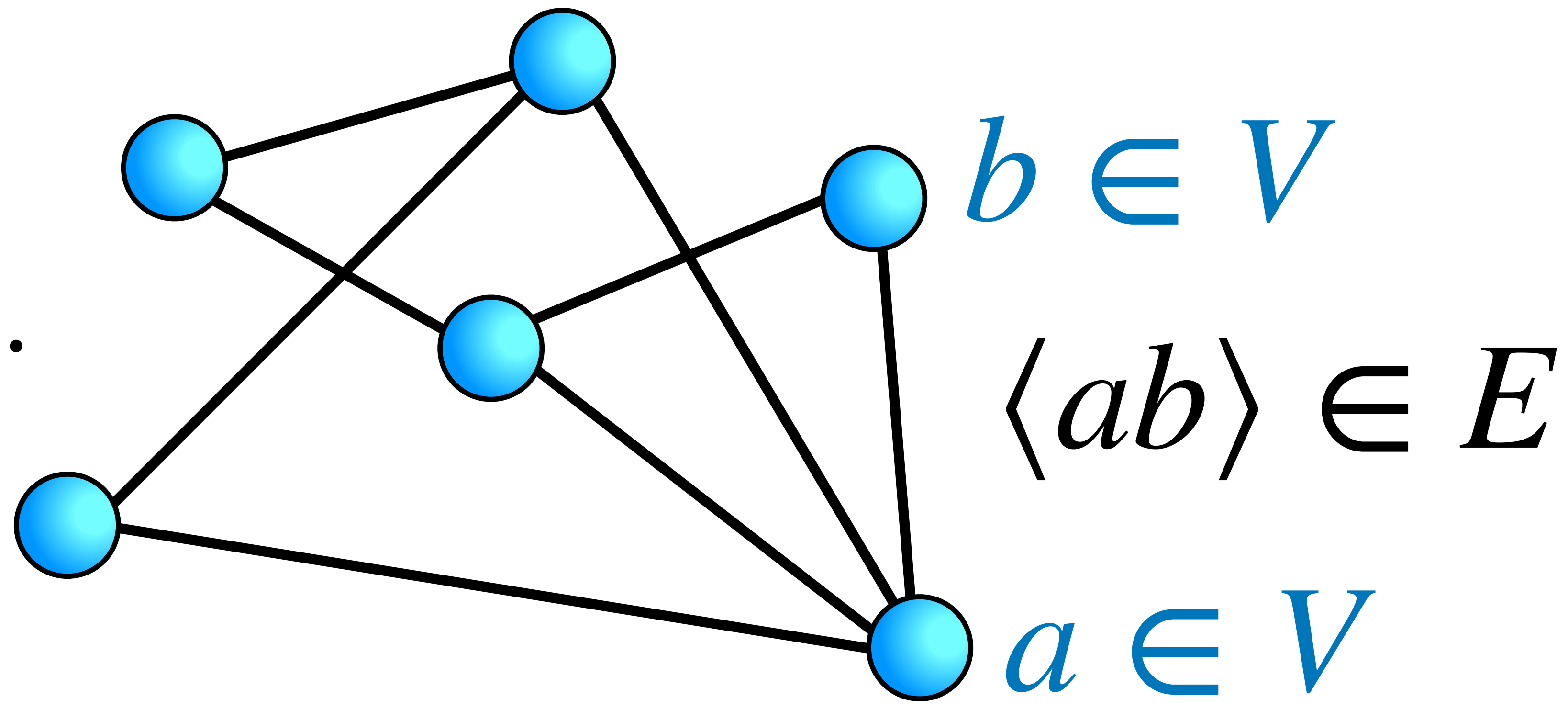
Graph state

There is a class of states generated by these ingredients, which are called *graph states*. [Hein et al. quant-ph/0602096]

- Graph = $\{V, E\}$
- V: vertices $\leftrightarrow$ qubits $|+\rangle^{\otimes V}$ are placed
- E: edges $\leftrightarrow$ $CZ_{a,b}$ is applied on $\langle ab \rangle \in E$ $(a, b \in V)$
- Graph state $\subset$ Stabilizer state
- Translationally invariant graph states are called *cluster states*.



Graph state

- In terms of state vectors,

$$|\psi_{\mathscr{C}}\rangle = \prod_{\langle vv'\rangle \in E} CZ_{v,v'} |+\rangle^{\otimes V}$$

- In terms of stabilizers,

$$|+\rangle^{\otimes V} \quad \longleftrightarrow \quad \left\{ X_v \,\middle|\, v \in V \right\}$$

$$|\psi_{\mathscr{C}}\rangle \quad \longleftrightarrow \quad \left\{ K_v \,\middle|\, v \in V \right\}$$

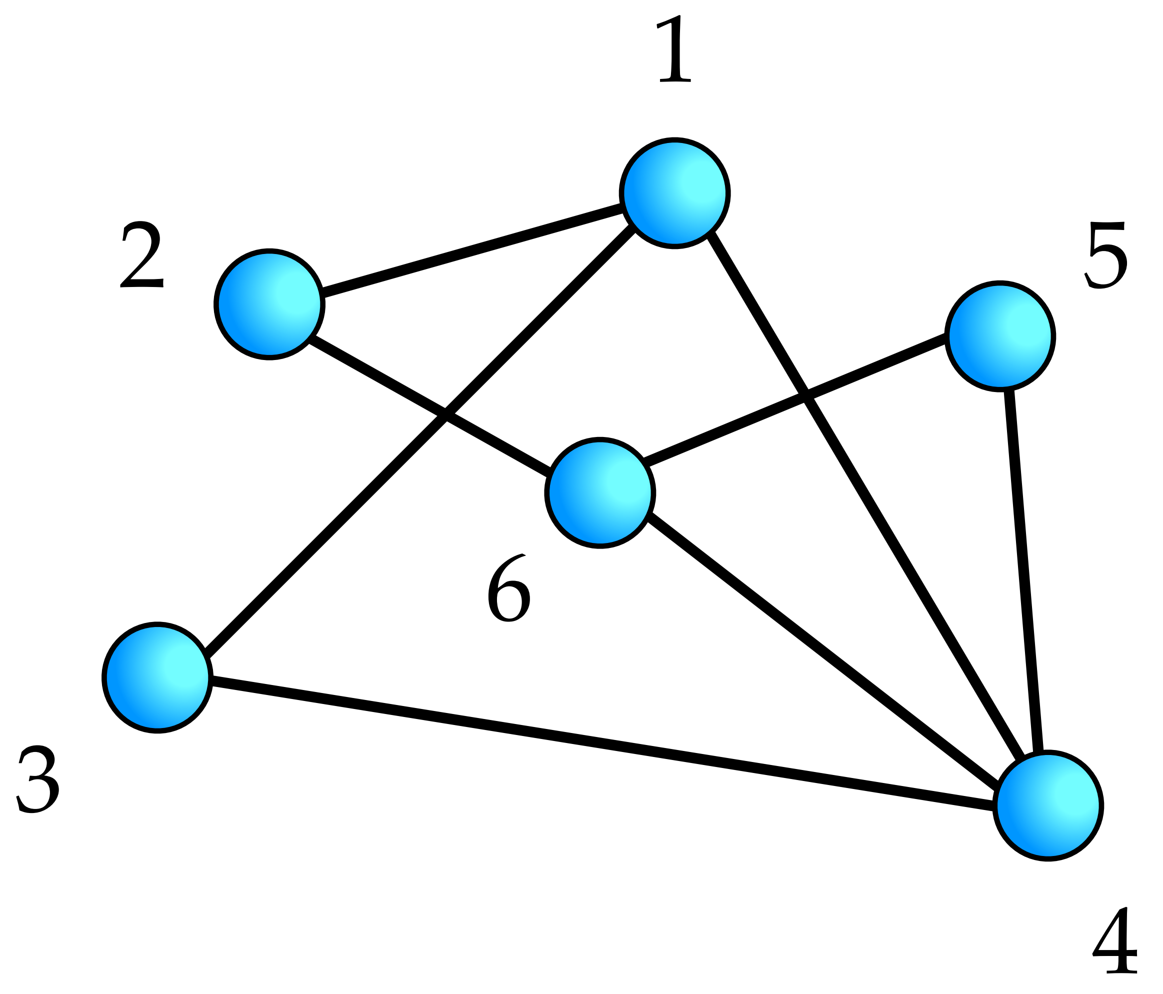
where

$$K_v = \Big(\prod_{\langle vv'\rangle \in E} CZ_{v,v'} \Big) \cdot X_v \cdot \Big(\prod_{\langle vv'\rangle \in E} CZ_{v,v'} \Big) = X_v \prod_{\langle vv'\rangle \in E} Z_{v'}$$

Graph state

$$K_1 = X_1 Z_2 Z_3 Z_4$$

$$K_2 = X_2 Z_1 Z_6$$

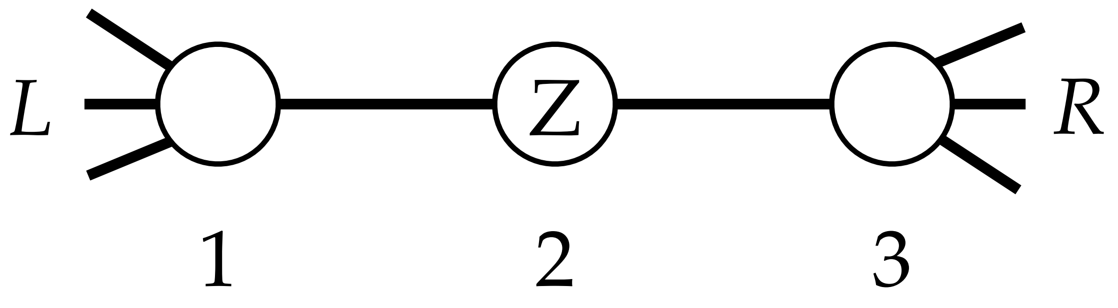


$$K_4 = X_3 Z_1 Z_4$$

etc.

Graph state

- Z measurement



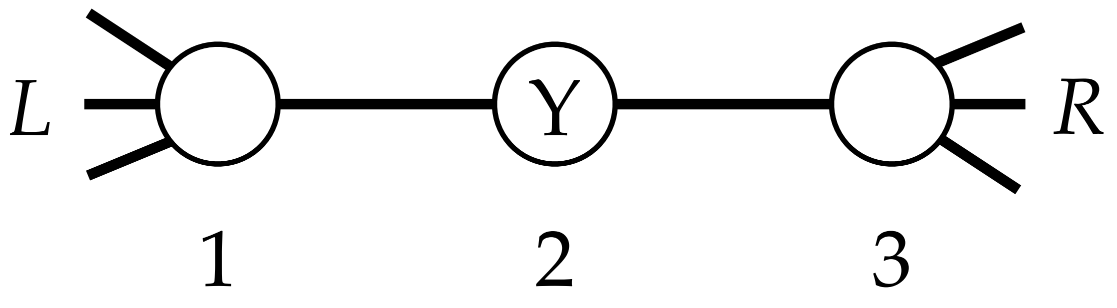
Stabilizers of the graph state:

$$K_1 = \prod_{j \in L} Z_j \cdot X_1 \underset{\pm 1}{Z_2}\,, \quad K_2 = Z_1 X_2 Z_3\,, \quad K_3 = \underset{\pm 1}{Z_2} X_3 \cdot \prod_{j \in R} Z_j$$

After the measurement:

$$K_1 = \prod_{j \in L} Z_j \cdot X_1 (\pm 1)\,, \quad K_3 = (\pm 1) X_3 \cdot \prod_{j \in R} Z_j$$

- Y measurement

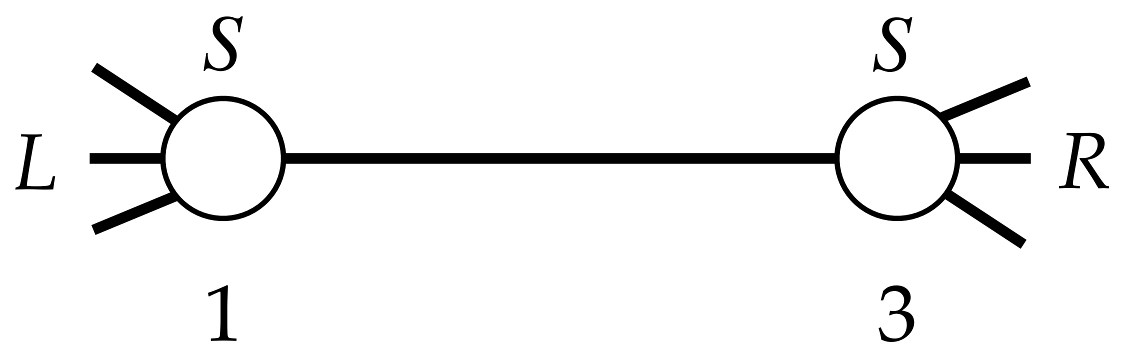


Stabilizers of the graph state:

$$K_1 = \prod_{j \in L} Z_j \cdot X_1 Z_2\,, \quad K_2 = Z_1 X_2 Z_3\,, \quad K_3 = Z_2 X_3 \cdot \prod_{j \in R} Z_j$$

Recombine:

$$K_1 K_2 = \prod_{j \in L} Z_j Y_1 \overset{\pm 1}{Y_2} Z_3\,, \quad K_2 K_3 = Z_1 \overset{\pm 1}{Y_2} Y_3 \prod_{j \in R} Z_j$$



$SX = Y$

Graph state

General rules:

See e.g. [Hein et al. quant-ph/0602096]

$$P^v_{z,\pm}|G\rangle = \frac{1}{\sqrt{2}}|z,\pm\rangle^v \otimes U^v_{z,\pm}|G - v\rangle$$

$$P^v_{y,\pm}|G\rangle = \frac{1}{\sqrt{2}}|y,\pm\rangle^v \otimes U^v_{y,\pm}|\tau_a(G) - v\rangle$$

$$P^v_{x,\pm}|G\rangle = \frac{1}{\sqrt{2}}|x,\pm\rangle^v \otimes U^v_{x,\pm}|\tau_{b_0}(\tau_a \circ \tau_{b_0}(G) - v)\rangle$$

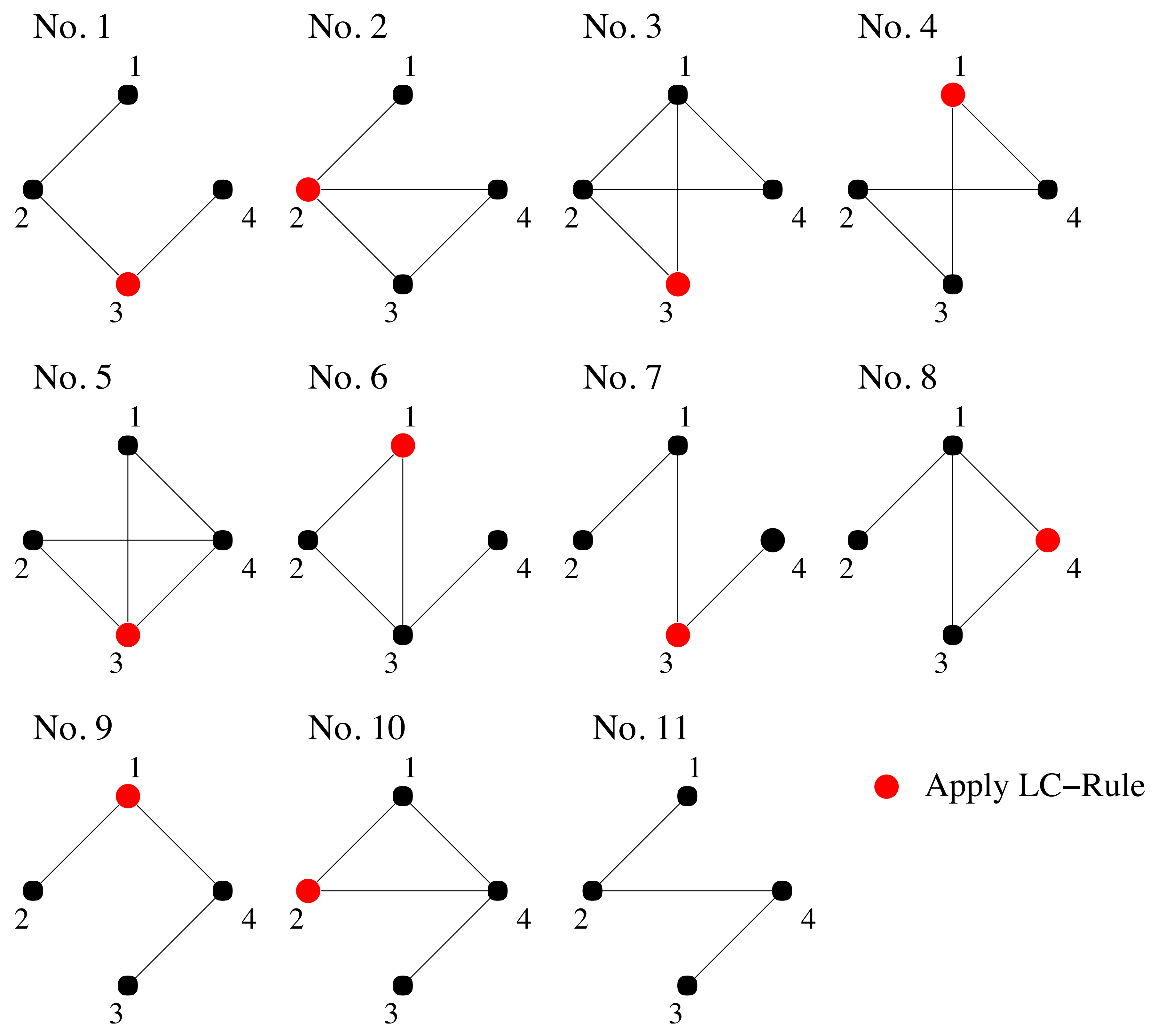
$\tau_a(G)$: local complementation of a in G.

b_0 : any choice from Nb(a)

$U^a_{x,y,z,\pm}$: outcome dependent ops. $\{Z, S, H\}$

We will use X measurement in part II, but we won't use the rule above.

Local complementation $\tau_a(G)$



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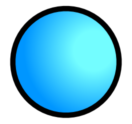
Gate teleportation

1-qubit state



$|\psi\rangle$

Gate teleportation

$|\psi\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$ $|+\rangle$

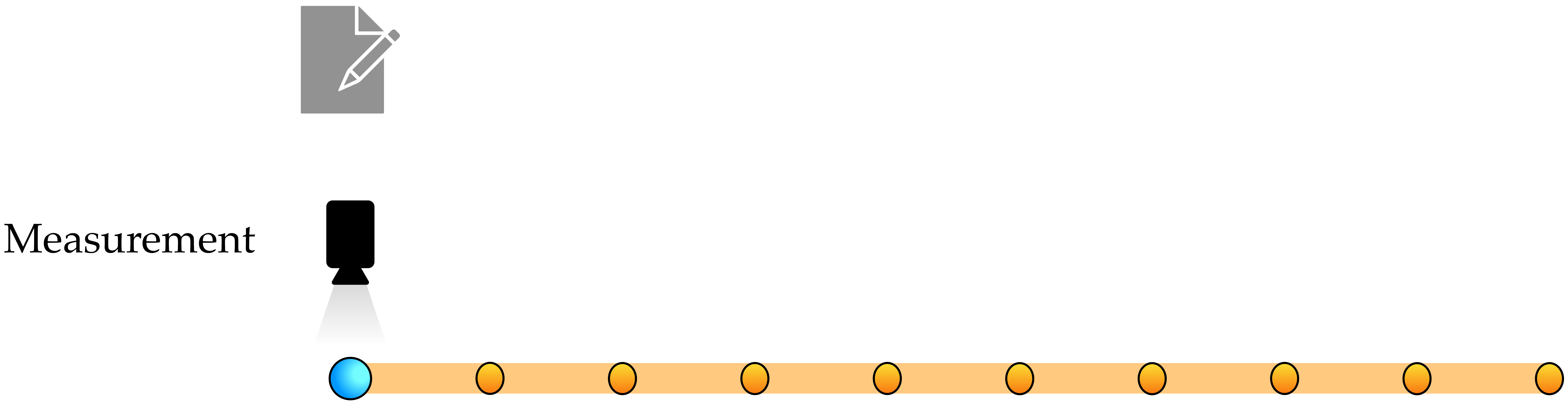
Gate teleportation

CZ gate

$|\psi\rangle$

1d *cluster state*

Gate teleportation

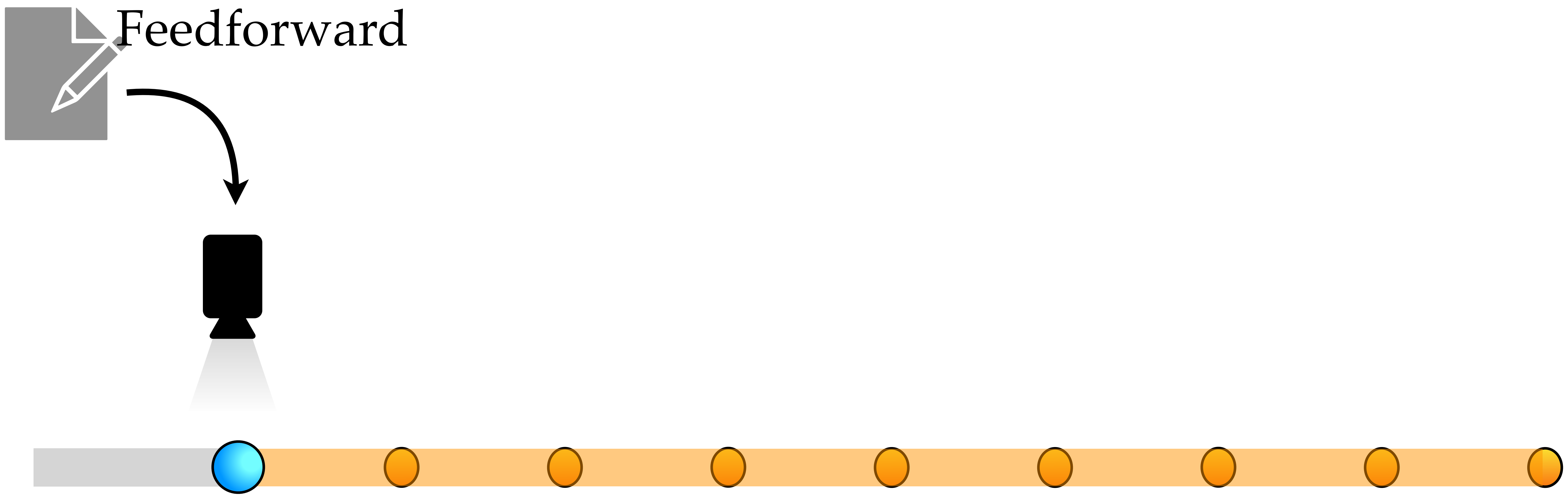


$|\psi\rangle$

Gate teleportation

$$X^{\#}Z^{\#} \cdot U_1 |\psi\rangle$$

Gate teleportation

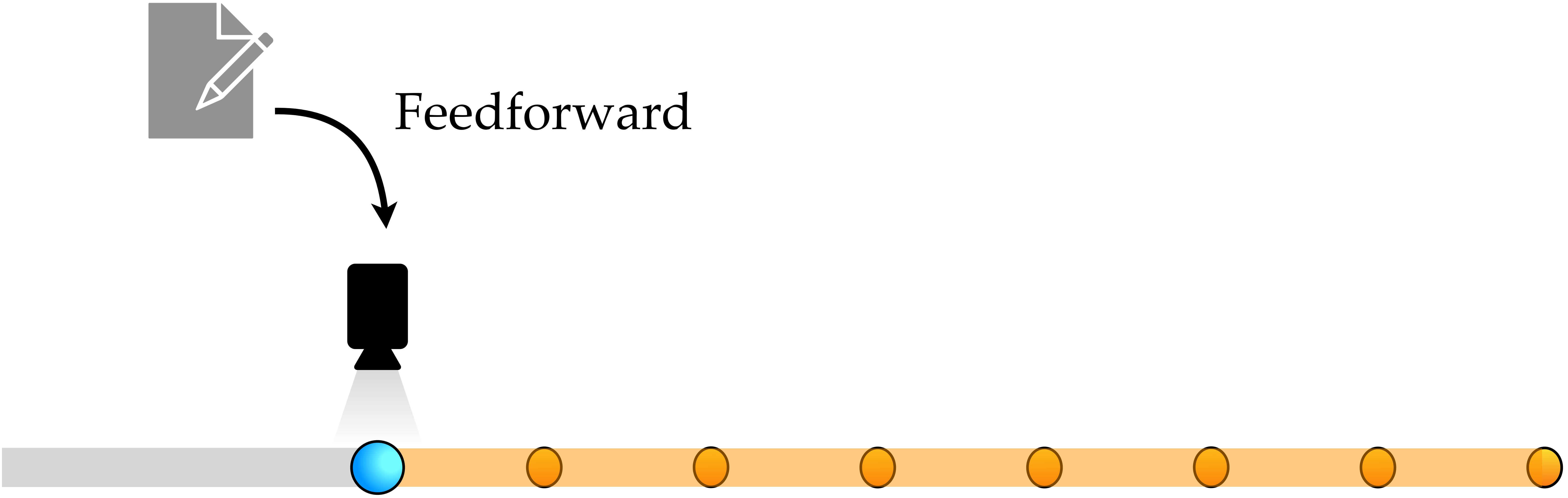


$$X^{\#}Z^{\#} \cdot U_1 |\psi\rangle$$

Gate teleportation

$$X^{\#}Z^{\#} \cdot U_2 U_1 |\psi\rangle$$

Gate teleportation



$$X^{\#}Z^{\#} \cdot U_2 U_1 |\psi\rangle$$

Gate teleportation

$$X^{\#}Z^{\#} \cdot U_3U_2U_1|\psi\rangle$$

Gate teleportation

Post-measurement product state

$$X^{\#}Z^{\#} \cdot U_N \cdots U_2 U_1 |\psi\rangle$$

Simulated state

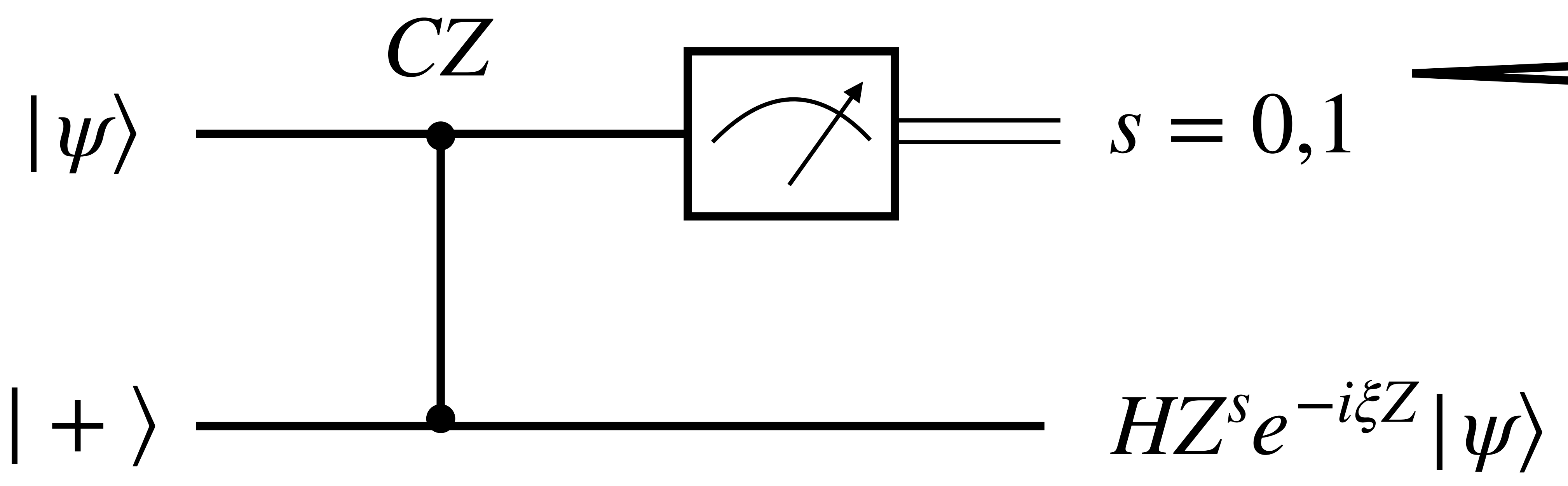
Gate teleportation

Post-measurement product state

$$U_N \cdots U_2 U_1 |\psi\rangle$$

Simulated state
(Post-processing)

Gate teleportation



$$\mathcal{M} = \left\{ e^{i\xi Z}|+\rangle, e^{i\xi Z}|-\rangle \right\} = \left\{ Z^s e^{i\xi Z}|+\rangle \,|\, s = 0,1 \right\}$$

This can be shown with simple algebras:

$$\langle +|_1 e^{-i\xi Z_1} Z_1^s \times \left(CZ_{1,2}|\psi\rangle_1 |+\rangle_2 \right)$$ **Inner product**

$$= \langle +|_1 CZ_{1,2} e^{-i\xi Z_1} Z_1^s |\psi\rangle_1 |+\rangle_2$$ $[CZ, Z] = 0$

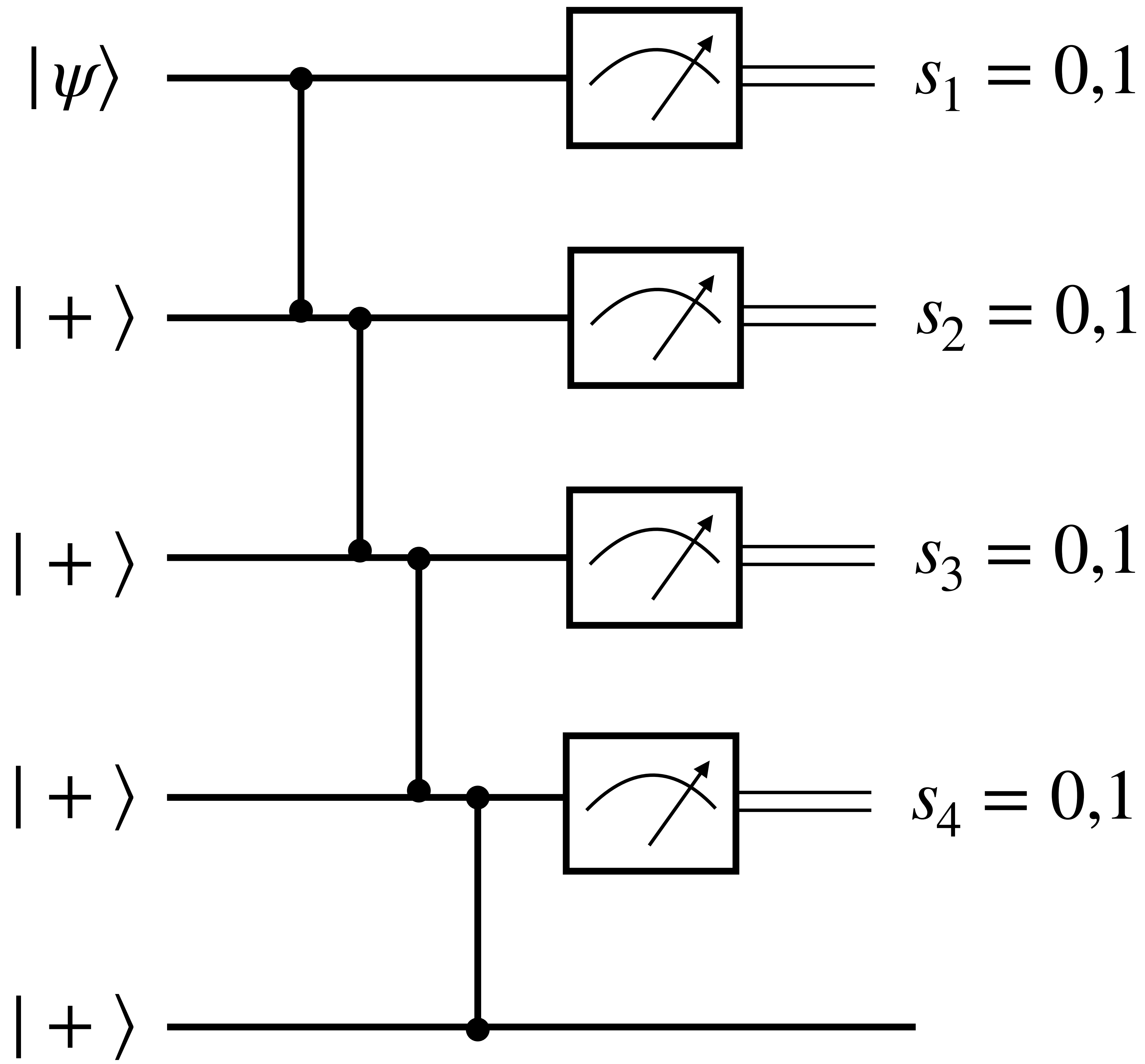
$$\sim \langle 0|_1 e^{-i\xi Z_1} Z_1^s |\psi\rangle_1 |+\rangle_2 + \langle 1|_1 e^{-i\xi Z_1} Z_1^s |\psi\rangle_1 Z_2 |+\rangle_2$$ $CZ_{1,2} = |0\rangle_1\langle 0|_1 \otimes I_2 + |1\rangle_1\langle 1|_1 \otimes Z_2$

$$= |+\rangle_2 \langle 0|_1 e^{-i\xi Z_1} Z_1^s |\psi\rangle_1 + |-\rangle_2 \langle 1|_1 e^{-i\xi Z_1} Z_1^s |\psi\rangle_1$$ $Z|+\rangle = |-\rangle$

$$= |+\rangle_2 \langle +|_1 H_1 e^{-i\xi Z_1} Z_1^s |\psi\rangle_1 + |-\rangle_2 \langle -|_1 H_1 e^{-i\xi Z_1} Z_1^s |\psi\rangle_1$$ $H|+\rangle = |0\rangle$ **and** $H|-\rangle = |1\rangle$

$$= H_2 e^{-i\xi Z_2} Z_2^s |\psi\rangle_2$$

Gate teleportation



The outcome state is applied by a cascade of unitary gates:

$$(HZ^{s_4}e^{-i\xi_4 Z})(HZ^{s_3}e^{-i\xi_3 Z})(HZ^{s_2}e^{-i\xi_2 Z})(HZ^{s_1}e^{-i\xi_1 Z})|\psi\rangle$$

Using $HZH = X$ and $XZ = -ZX$, we get

$$(X^{s_4}e^{-i\xi_4 X})(Z^{s_3}e^{-i\xi_3 Z})(X^{s_2}e^{-i\xi_2 X})(Z^{s_1}e^{-i\xi_1 Z})|\psi\rangle$$

$$= X^{s_4+s_2}Z^{s_3+s_1}e^{-i\xi_4(-1)^{s_1+s_3}X}e^{-i\xi_3(-1)^{s_2}Z}e^{-i\xi_2(-1)^{s_1}X}e^{-i\xi_1 Z}|\psi\rangle\,.$$

If we set $\xi_1 = 0,\ \xi_2 = (-1)^{s_1}\gamma\,,\ \xi_3 = (-1)^{s_2}\beta\,,$

$\xi_4 = (-1)^{s_1+s_3}\alpha$, the output state becomes

$$X^{s_4+s_2}Z^{s_3+s_1}e^{-i\alpha X}e^{-i\beta Z}e^{-i\gamma X}|\psi\rangle$$

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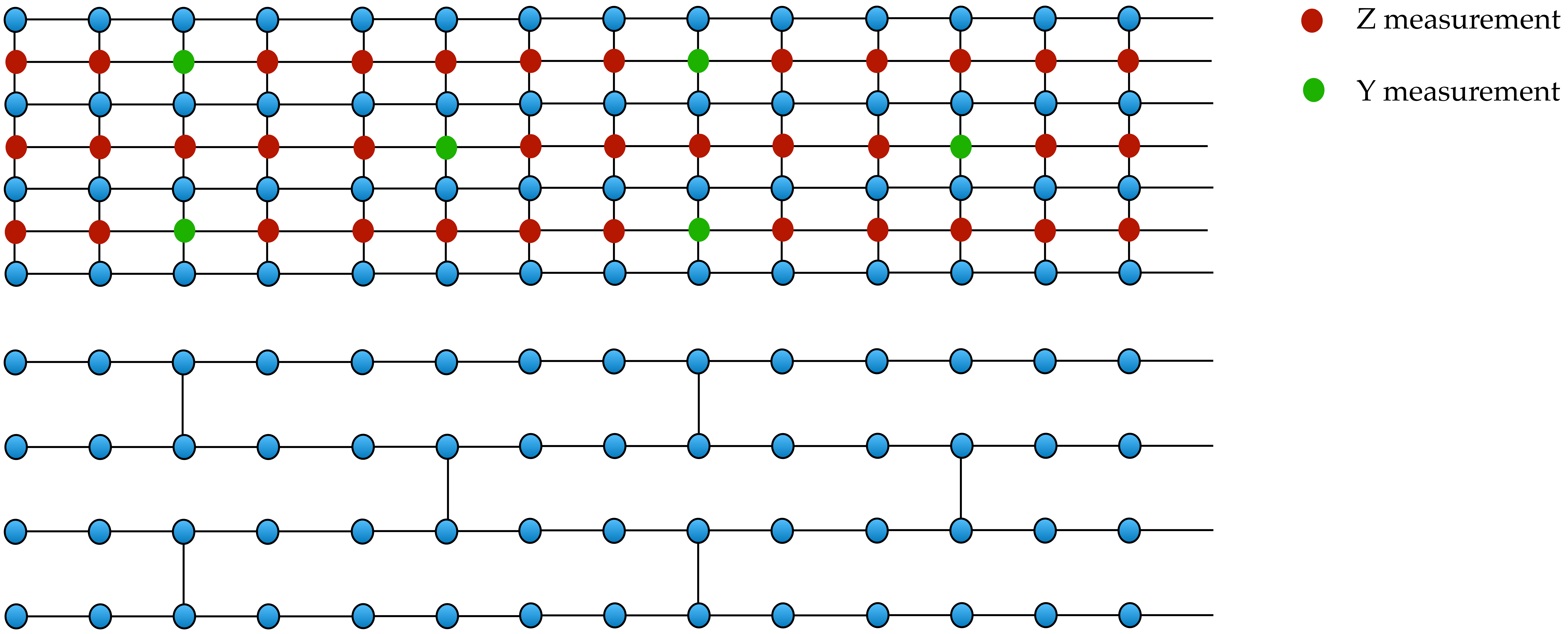
2d cluster state on square lattice is universal

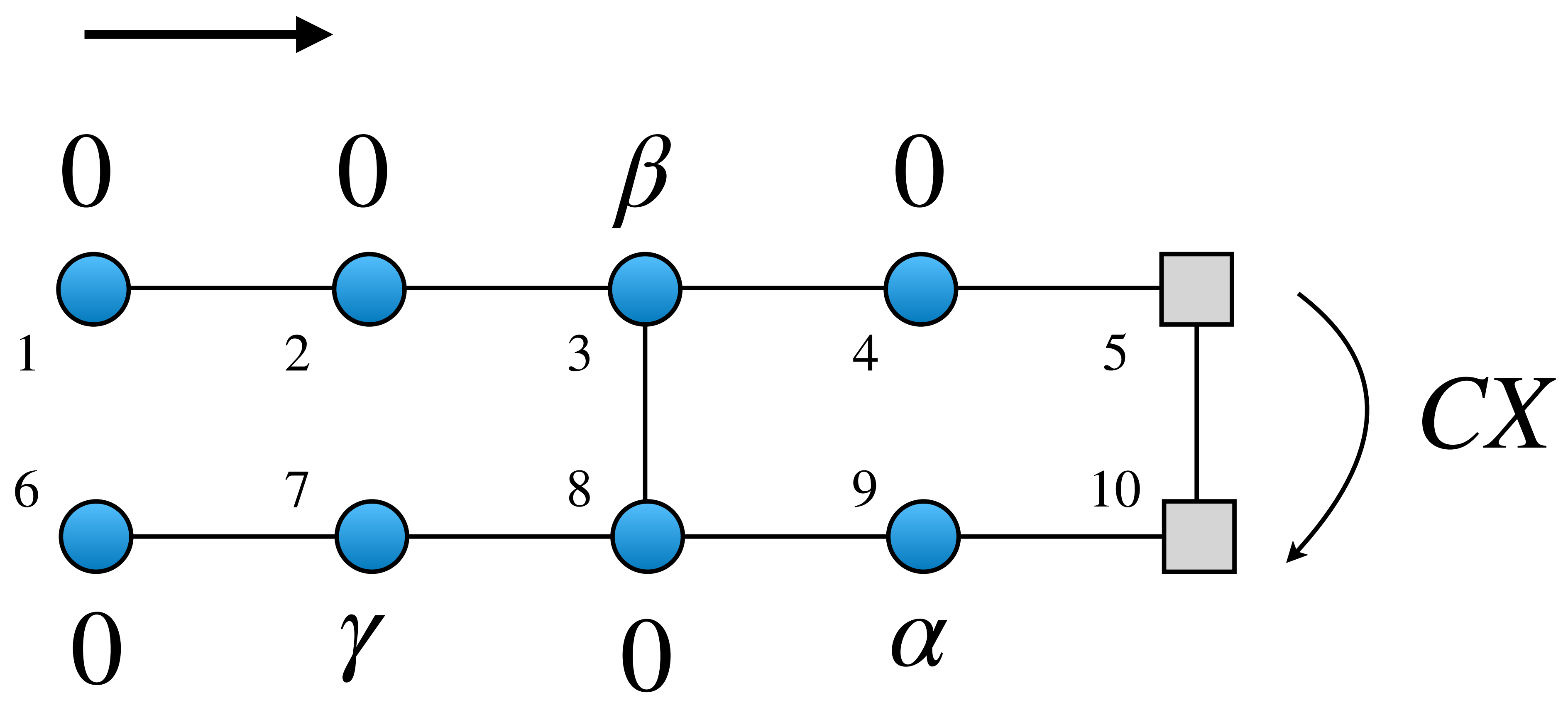
From a square-lattice graph state to a brickwork graph state.



2d cluster state on square lattice is universal

CNOT gate by measuring the brickwork graph state.

Measurement basis: $\{e^{i\xi Z}|+\rangle, e^{i\xi Z}|-\rangle\}$.



The state at 5 & 10 ($\mathscr{H}_5 \otimes \mathscr{H}_{10}$) gets the following unitary

$$CZ\,(HZ^{s_4} \otimes He^{i\alpha Z}Z^{s_9})\,(He^{i\beta Z}Z^{s_3} \otimes HZ^{s_8})$$
$$\times\, CZ\,(HZ^{s_2} \otimes He^{i\gamma Z}Z^{s_7})\,(HZ^{s_1} \otimes HZ^{s_6})$$

It is equal to (a good exercise to check):

$$CZ\,(X^{s_4} \otimes e^{i\alpha X}X^{s_9})\,(e^{i\beta Z}Z^{s_3} \otimes Z^{s_8})$$
$$\times\, CZ\,(X^{s_2} \otimes e^{i\gamma X}X^{s_7})\,(Z^{s_1} \otimes Z^{s_6})$$
$$= \pm\,(X^{s_2+s_4}Z^{s_1+s_3+s_9} \otimes X^{s_7+s_9}Z^{s_4+s_6+s_8})$$
$$\times \exp[i(-1)^{s_2}\beta Z \otimes I]\exp[i(-1)^{s_2+s_6+s_8}\alpha Z \otimes X]$$
$$\times \exp[i(-1)^{s_6}\gamma I \otimes X]$$

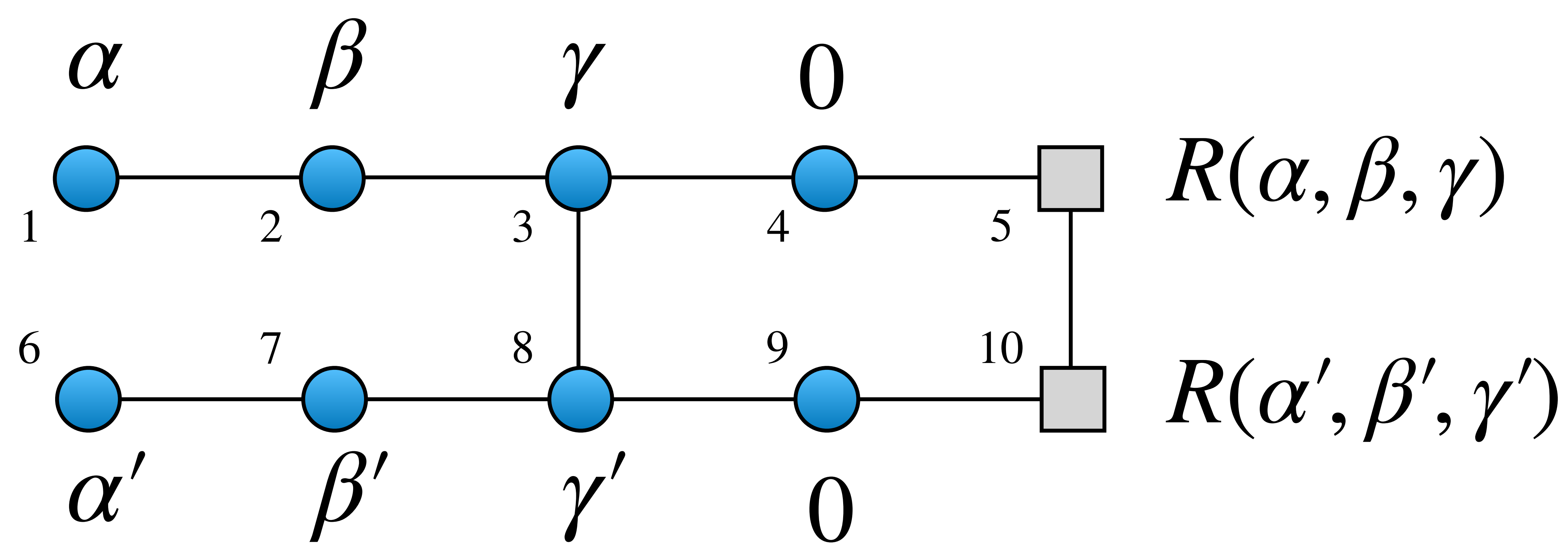
Setting the parameters as $\alpha = (-1)^{s_2+s_6+s_8} \times \dfrac{-\pi}{4}$, $\beta = (-1)^{s_2} \times \dfrac{\pi}{4}$, $\gamma = (-1)^{s_6} \times \dfrac{\pi}{4}$, we obtain

$$\exp\Big[\frac{-i\pi}{4}(I - Z_5)(I - X_{10})\Big] = CX_{5,10}\,.$$

2d cluster state on square lattice is universal

SU(2) rotation by measuring the brickwork graph state.

Measurement basis: $\{e^{i\xi Z}|+\rangle, e^{i\xi Z}|-\rangle\}$.



Similarly, the measurement pattern in the left figure gives us the Euler rotation.

$$CZ\,(HZ^{s_4} \otimes HZ^{s_9})\,(HZ^{s_3}e^{i\gamma Z} \otimes HZ^{s_8}e^{i\gamma' Z})CZ$$
$$\times\,(HZ^{s_2}e^{i\beta Z} \otimes HZ^{s_7}e^{i\beta' Z})\,(He^{i\alpha Z}Z^{s_1} \otimes HZ^{s_6}e^{i\alpha' Z})$$

Cleaning up the above expression gives us

$$R(\alpha, \beta, \gamma) \otimes R(\alpha', \beta', \gamma')$$

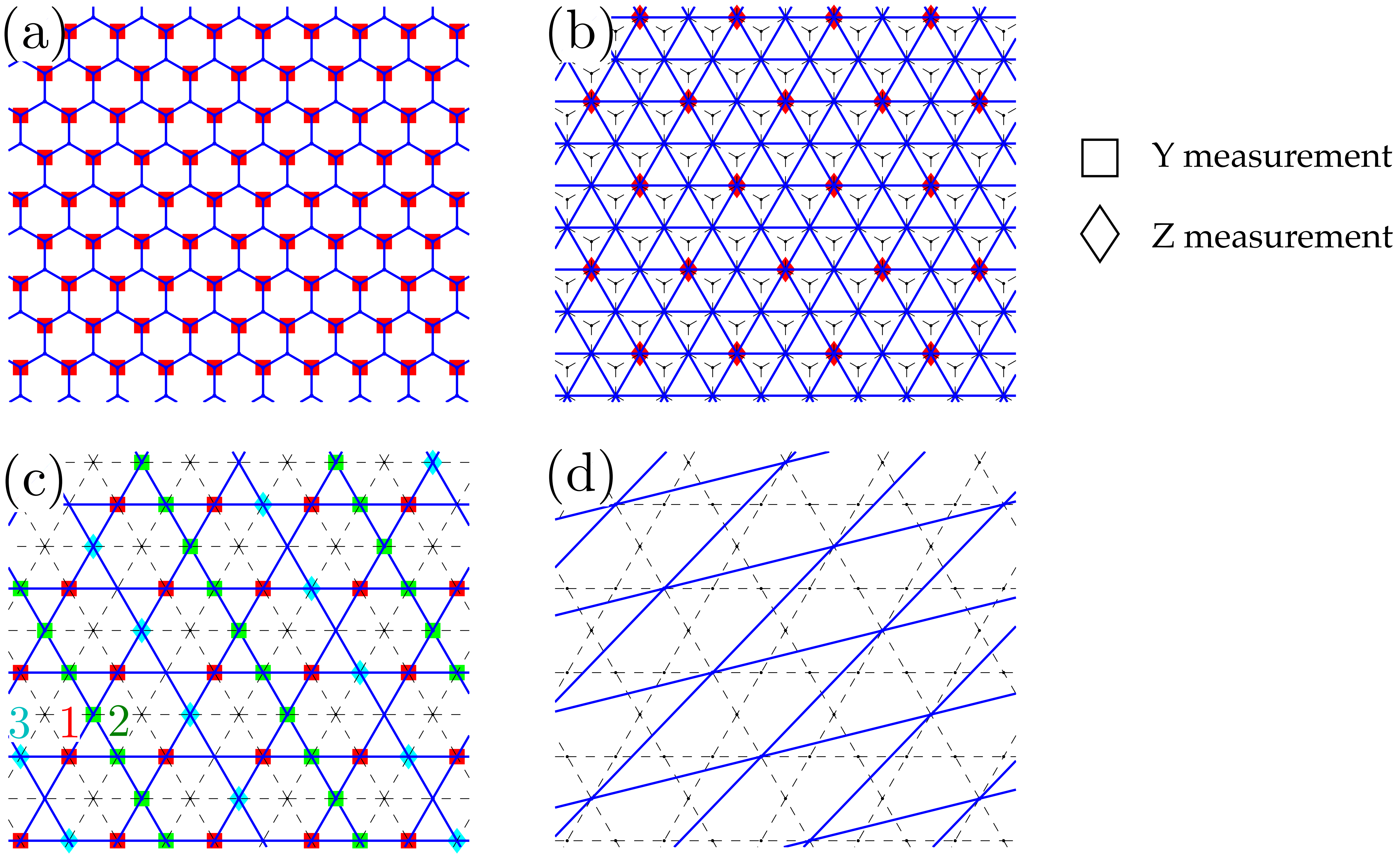
up to byproduct operators.

Therefore, the brickwork state is a universal resource of MBQC.

Cf. This state also has an application in “blind quantum computation” [Broadbent et al. quant-ph/0807.4154]

2d cluster state on square lattice is universal

Indeed, a graph states on any 2d regular lattice can be converted to the square-lattice graph state by measurement.



[Van den Nest et al. quant-ph/0604010]

MBQC

What we have just shown is a simple example of MBQC.

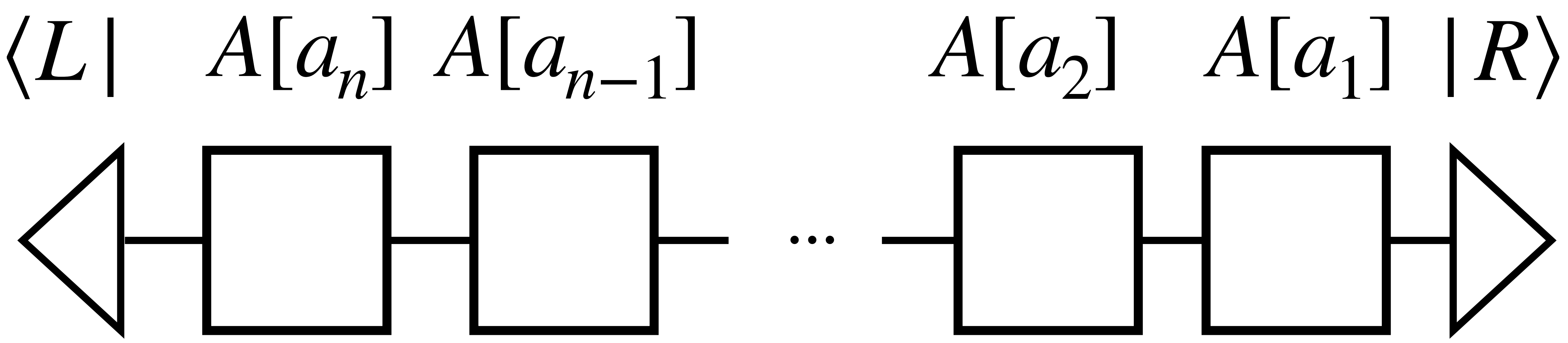
MBQC (measurement-based quantum computation)

(Universal) quantum computation can be achieved by
(1) **preparing a resource state**
(2) **measuring the resource state in a certain adaptive pattern.**
(3) **post-processing (unwanted) byproduct operators**

[Raussendorf-Briegel (2001)]
Review article: e.g. [T.-C. Wei (2023)]

MBQC in edge modes of 1d resource state

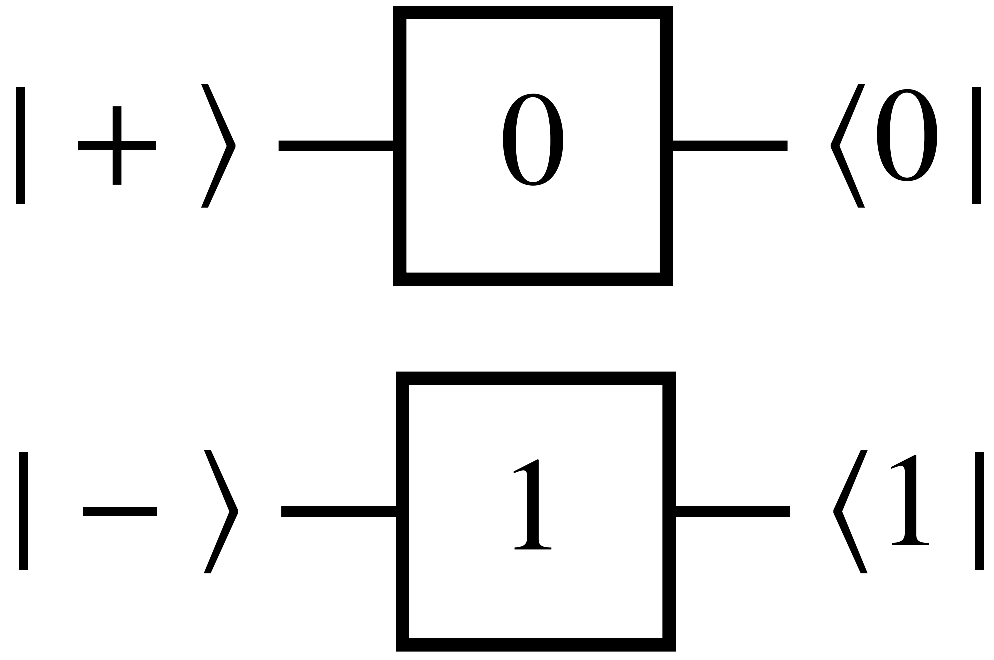
MPS representation of the 1d graph state (also called the 1d cluster state)

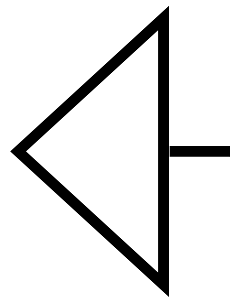


$$|\psi_{\mathscr{C}}\rangle = \sum_{\{a_k\}_{k=1,\ldots,n}} \langle L|A[a_n]A[a_{n-1}]\cdots A[a_2]A[a_1]|R\rangle \times |a_1, a_2, \ldots\rangle\rangle$$

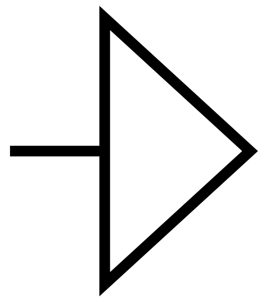
Virtual space **Physical qubits**

$A[a]$





$\langle L| = \langle 0|$

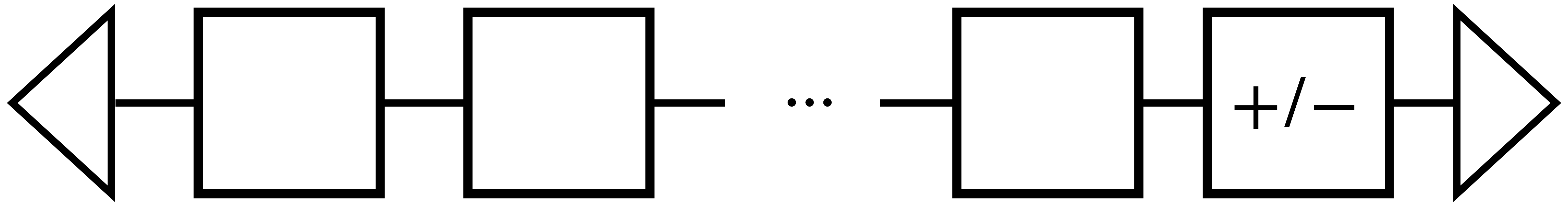


$|R\rangle = |+\rangle$ or an arbitrary edge state $|\phi\rangle$

MBQC in edge modes of 1d resource state

Measure the 1st qubit in the X basis: $\frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^s|1\rangle\Big)$

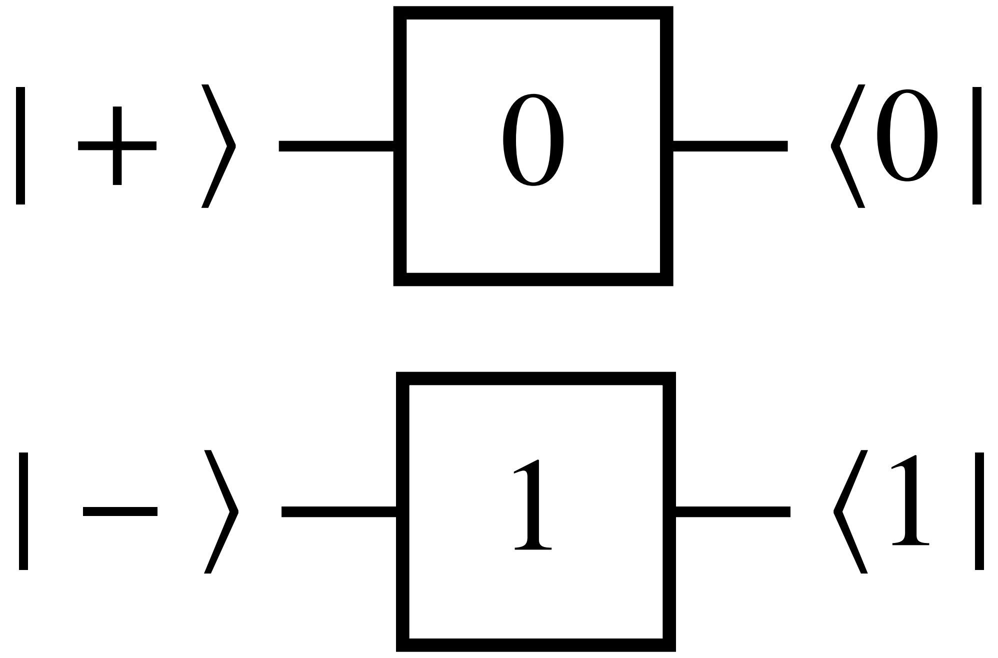
$\langle L|$ $A[a_n]$ $A[a_{n-1}]$ $A[a_2]$ $|R\rangle$

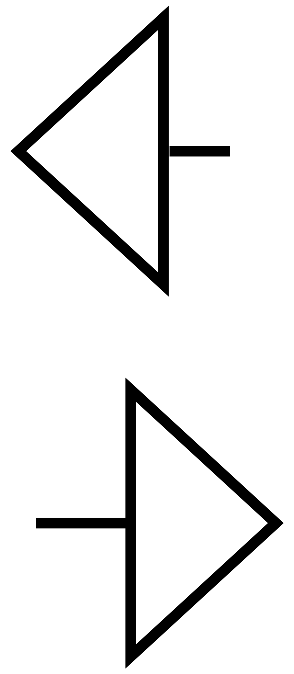


$$\sum_{\{a_k\}_{k=2,\ldots,n}} \langle L|A[a_n]A[a_{n-1}]\cdots A[a_2]\Big(A[0] + (-1)^s A[1]\Big)|R\rangle \times |s\rangle\rangle_1^{(X)}|a_2,\ldots\rangle\rangle$$

$HZ^s|R\rangle = |R_1\rangle$

$A[a]$





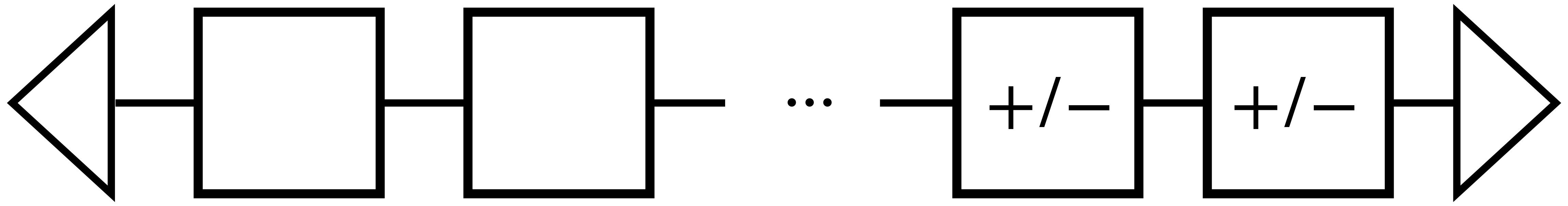
$\langle L| = \langle 0|$

$|R\rangle = |+\rangle$ or an arbitrary edge state $|\phi\rangle$

MBQC in edge modes of 1d resource state

Measure the 2nd qubit in the X basis: $\frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^s|1\rangle\Big)$

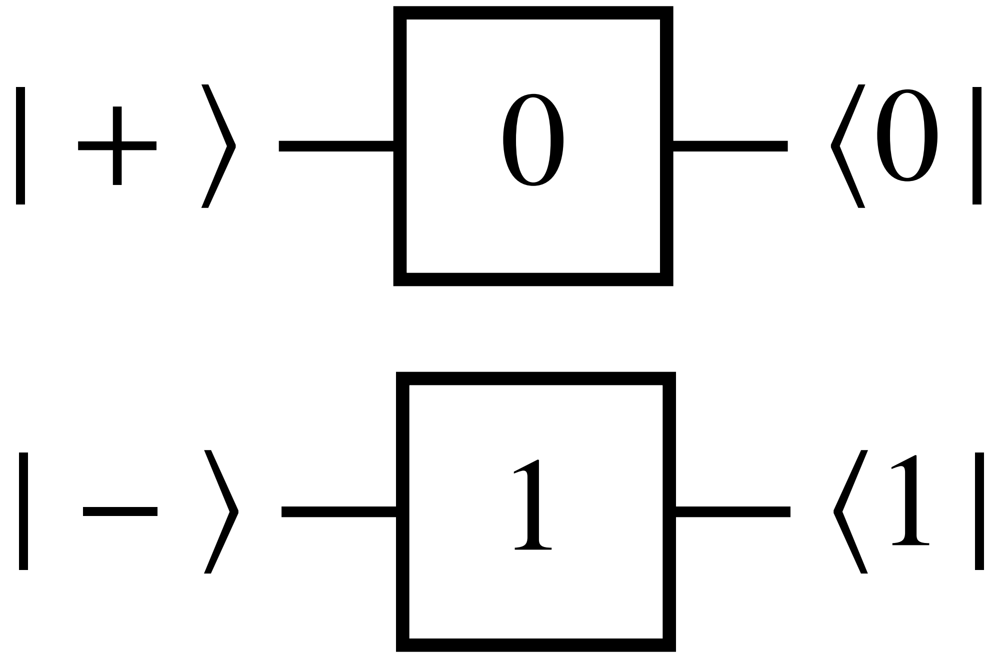
$\langle L|$ $A[a_n]$ $A[a_{n-1}]$ $|R\rangle$

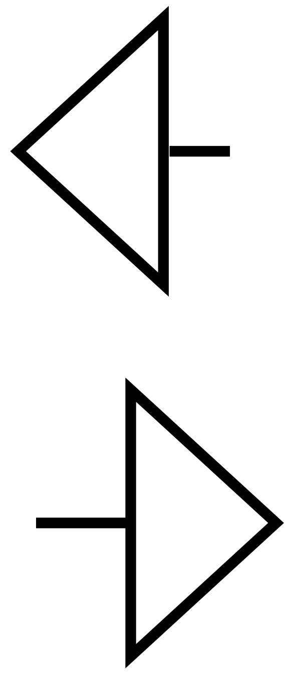


$$\sum_{\{a_k\}_{k=2,\dots,n}} \langle L|A[a_n]A[a_{n-1}]\cdots\Big(A[0]+(-1)^s A[1]\Big)|R_1\rangle \times |s\rangle\!\rangle_1^{(X)}|s\rangle\!\rangle_2^{(X)}|\dots\rangle\!\rangle$$

$$HZ^s|R_1\rangle = |R_2\rangle$$

$A[a]$



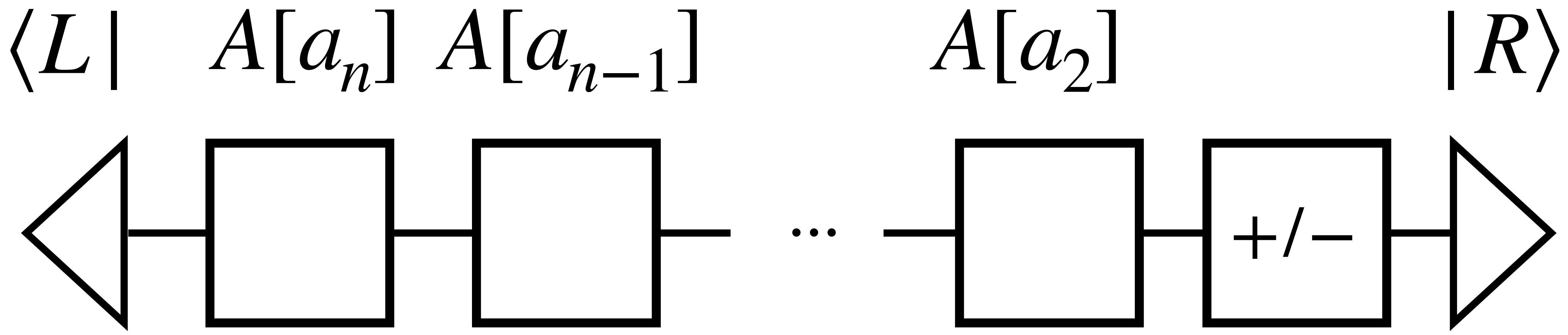


$\langle L| = \langle 0|$

$|R\rangle = |+\rangle$ or an arbitrary edge state $|\phi\rangle$

MBQC in edge modes of 1d resource state

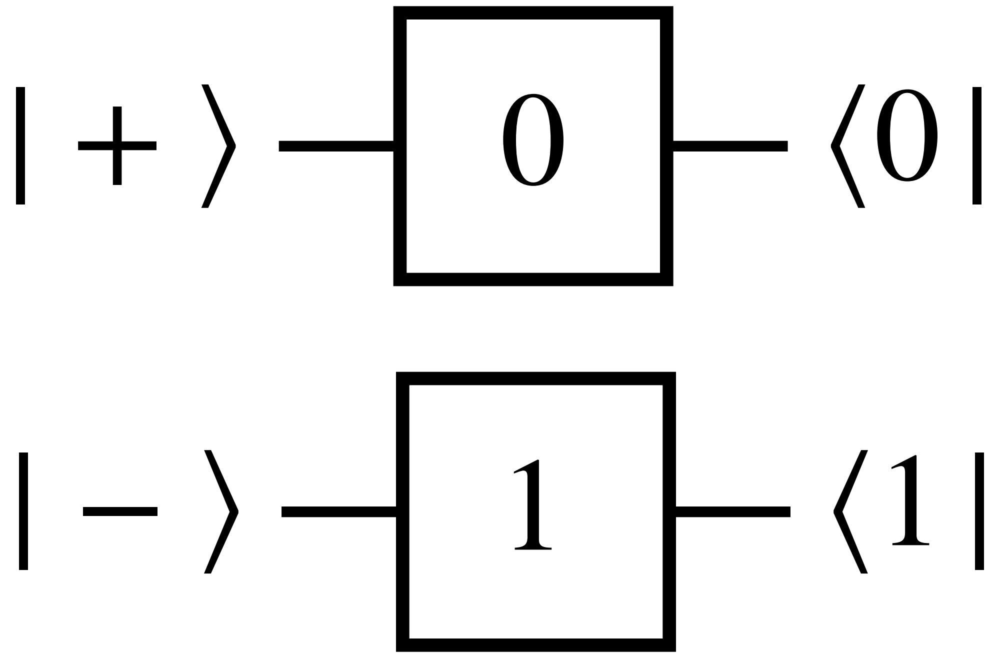
Measure the 1st qubit in the X basis: $\frac{1}{\sqrt{2}}\left(e^{i\theta}|0\rangle + (-1)^s e^{-i\theta}|1\rangle\right)$

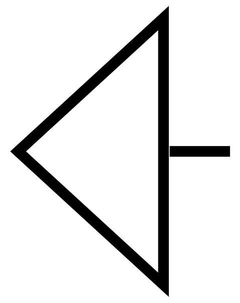


$$\sum_{\{a_k\}_{k=2,\ldots,n}} \langle L|A[a_n]A[a_{n-1}]\cdots A[a_2]\left(e^{-i\theta}A[0] + (-1)^s e^{i\theta}A[1]\right)|R\rangle \times |s\rangle\rangle_1^{(X)}|a_2,\ldots\rangle\rangle$$

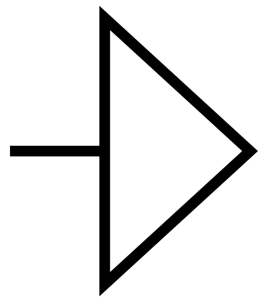
$HZ^s e^{-i\theta Z}|R\rangle = |R_1\rangle$

$A[a]$





$\langle L| = \langle 0|$

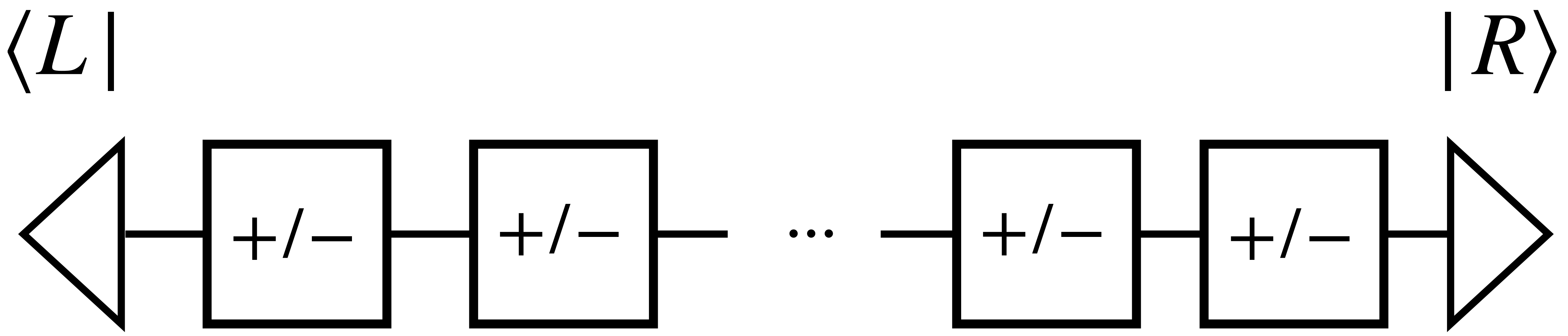


$|R\rangle = |+\rangle$ or an arbitrary edge state $|\phi\rangle$

MBQC in edge modes of 1d resource state

We have unitary gates acting on the virtual space $U_k \in \{HZe^{-i\theta_k Z}\}$

$\langle L| = \langle 0|$



$|R\rangle = |\phi\rangle$

$$\langle L| U_n U_{n-1} \cdots U_2 U_1 |R\rangle \times |s_1\rangle\rangle_1^{(X)} |s_2\rangle\rangle_2^{(X)} \cdots$$

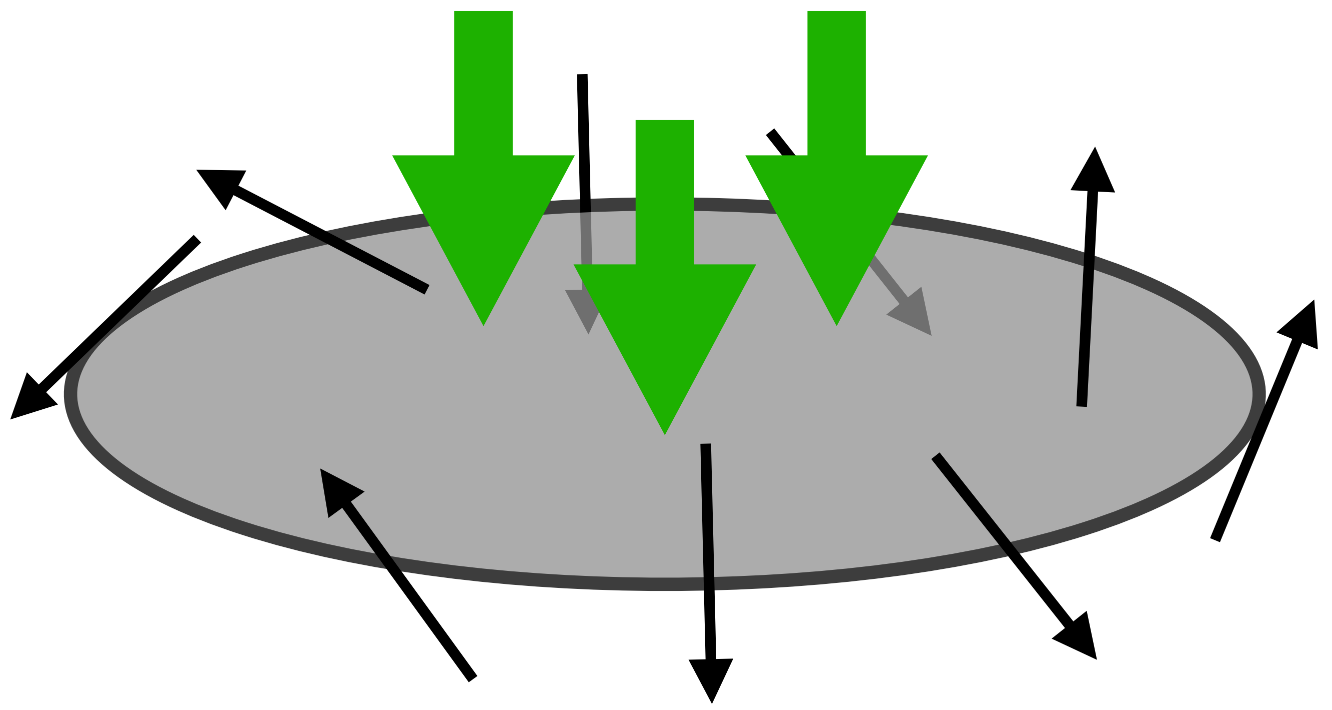
In the virtual space, we get quantum gates that generates $SU(2)$ rotations on an "initial state" $|\phi\rangle$,

$$U_n U_{n-1} \cdots U_2 U_1 |R\rangle$$

Once we measure all the physical qubits, we observe the probability distribution of projecting the virtual state to $|L\rangle$.

MBQC in edge modes of 1d resource state

Edge modes seem to play an important role in MBQC. [Gross-Eisert (2006)]



Indeed, resource states for the universal MBQC found so far belong to some SPT phases, states in which admit degenerate boundary modes.

E.g. AKLT state, cluster states in 1d/2d.

Some works have even proved that the universal MBQC is possible with states in the entire SPT phase. E.g. 2d cluster phase (protected by rigid line symmetries.)
[Raussendorf-Okay-Wang-Stephen-Nautrup 2018]

Plan

Part I: Quantum computation by measurement

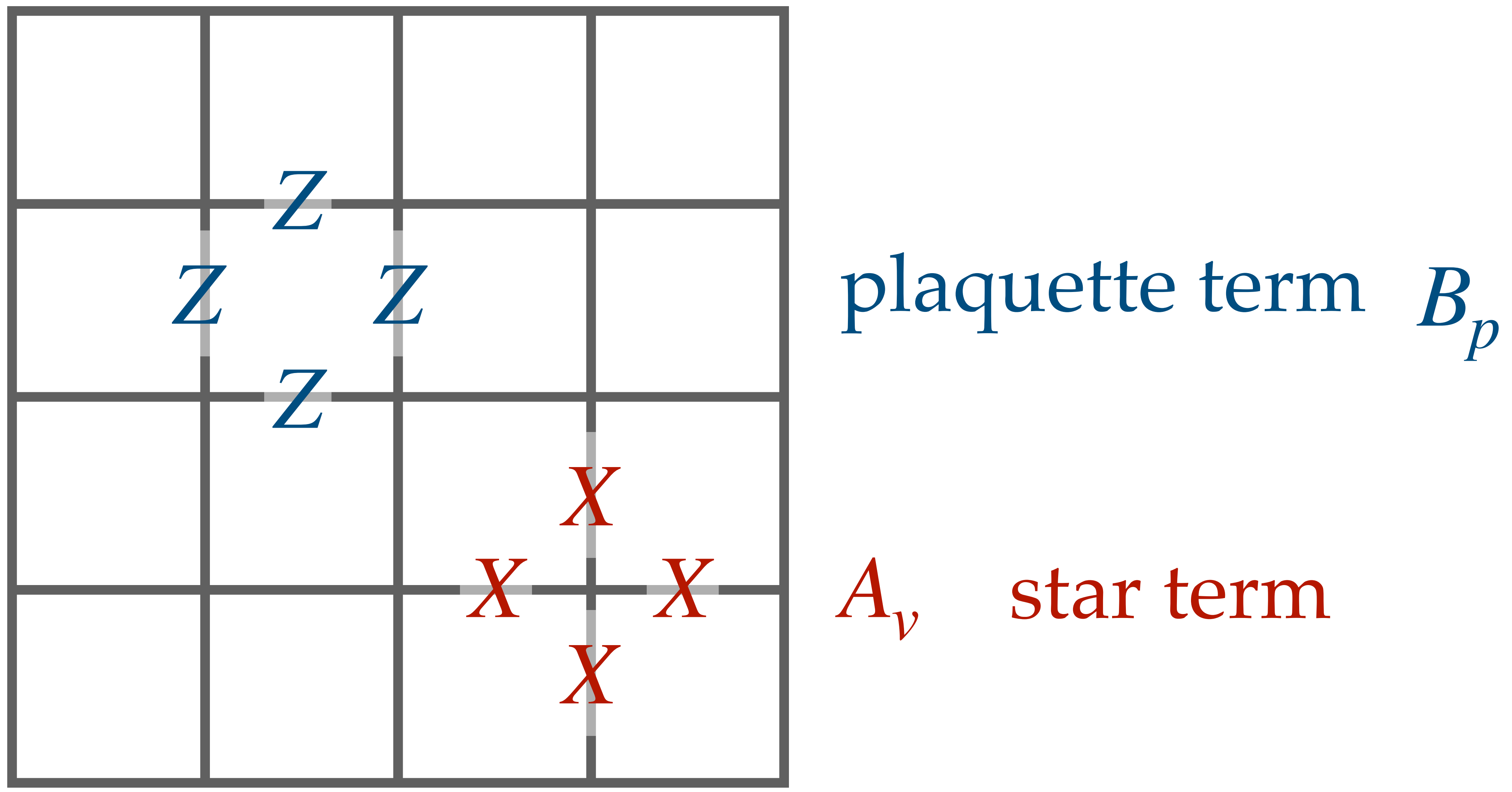
- Stabilizer formalism
- Graph state
- Gate teleportation
- Universal quantum computation on graph states

Part II: Measurement-based quantum computation and lattice gauge theory

- Measurement as a shortcut to topological orders
- $\mathbb{Z}_2$ lattice gauge theory
- Quantum simulation of lattice gauge theories

Toric code

- Kitaev's toric code
- Described by a Hamiltonian
$$H_{\mathrm{TC}} = -\sum_v A_v - \sum_p B_p$$
- $A_v|gs\rangle = B_p|gs\rangle = |gs\rangle$.
- $\#$ edges $= 2|V|$
- $\#$ plaquettes $= |V|$
- $\#$ vertices $= |V|$
- On a torus, stabilizers are not completely independent:
$$\prod_{p\in P} B_p = 1\,, \quad \prod_{v\in V} A_v = 1\,.$$



The ground state is degenerate, and the degeneracy depends on the background topology.

→ Topological order.

Long-range entanglement

- Bravyi-Hastings-Verstraete (2006) showed that ground states with a topological order cannot be prepared by any local time-dependent Hamiltonian evolution from any product state within a finite time.
- Finite-time (finite depth of quantum circuits) : $\mathcal{O}(1)$ with respect to the system size.
- In condensed matter physics, this is used to classify different topological orders of gapped quantum systems. $\rightarrow$ Long-range entanglement

Gapped ground states with different topological orders cannot be connected by finite-depth local unitary transformations.

- *The toric code state is a long-range entangled state.*

Short-range entanglement

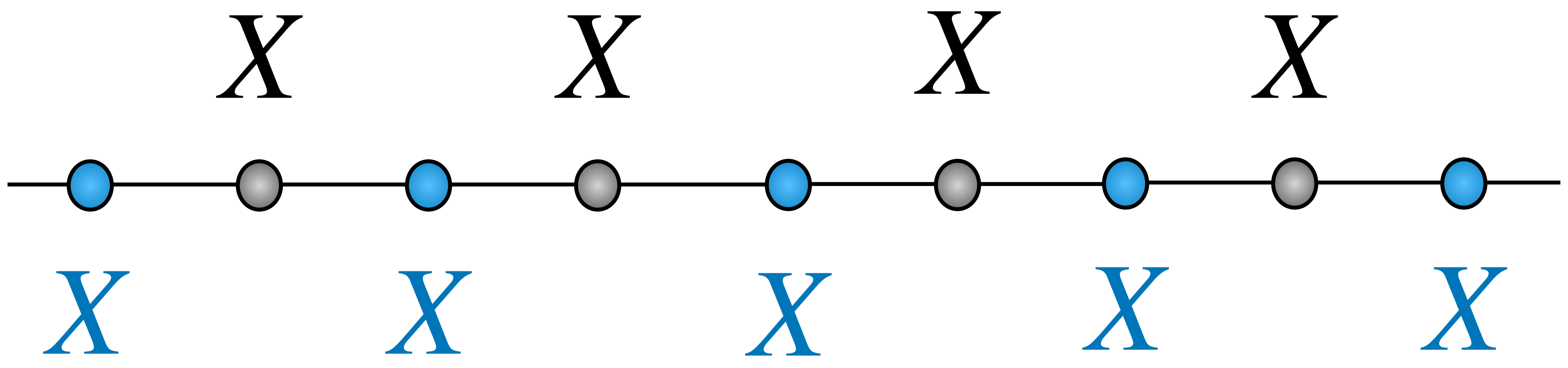
- When a system is not long-range entangled, it is said to be short-range entangled.
- Are short-range entangled states uninteresting?
- There are states that cannot be obtained by finite-depth local symmetry-preserving unitary transformations.
- They are called Symmetry-Protected Topological order states.

SPT-ordered states cannot be prepared from a product state by finite-depth symmetry-preserving local unitary transformations.

- Note, however, that if you wish to prepare an SPT ordered state, you can simply construct a finite-depth local unitary circuit without symmetries.
- *Cluster states are short-range entangled states.*

Short-range entanglement

- 1d cluster state is an SPT protected by $\mathbb{Z}_2[0] \times \mathbb{Z}_2[0]$



$$1 = \prod_{j\in\mathbb{Z}} K_{2j} = \prod_{j\in\mathbb{Z}} Z_{2j-1}X_{2j}Z_{2j+1} = \prod_{j\in\mathbb{Z}} X_{2j}$$

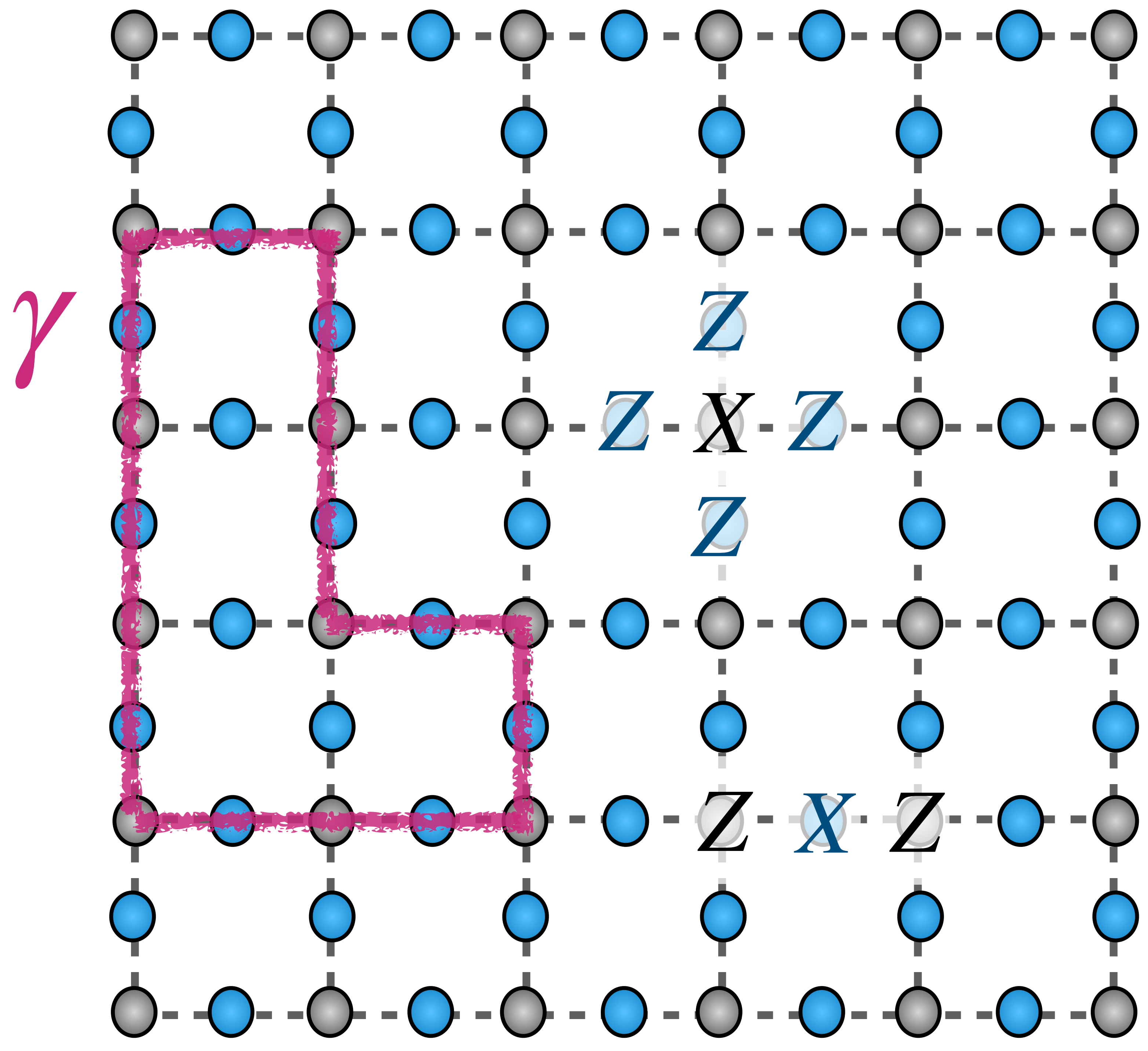
$$1 = \prod_{j\in\mathbb{Z}} K_{2j+1} = \prod_{j\in\mathbb{Z}} Z_{2j}X_{2j+1}Z_{2j+2} = \prod_{j\in\mathbb{Z}} X_{2j+1}$$

$[CZ, \prod_{\text{even}} X] \neq 0, \quad [CZ, \prod_{\text{odd}} X] \neq 0$, thus we cannot use CZ as a symmetry-preserving local unitary to bring it down to the trivial product state.

Short-range entanglement

- 2d cluster state protected by $\mathbb{Z}_2[0] \times \mathbb{Z}_2[1]$

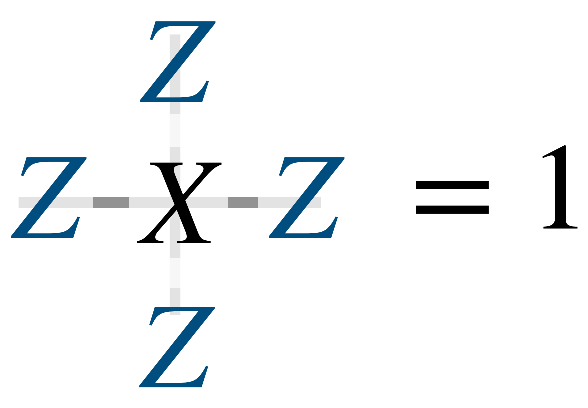
e.g. [Yoshida (2016)] [HS-Okuda (2022)] [Verresen-Borla-Vishwanath-Moroz-Thorngren (2022)]



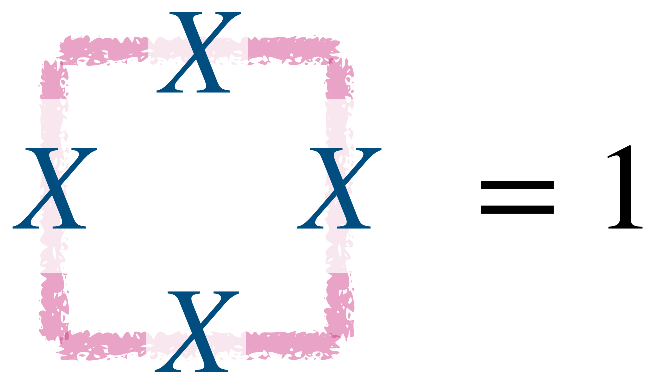
$$1 = \prod_v K_v = \prod_v X_v \quad : \mathbb{Z}_2[0]$$

$$1 = \prod_{e\in\gamma} K_e = \prod_{e\in\gamma} X_e \quad : \mathbb{Z}_2[1]$$

Note some similarity with the toric code, although they are in different phases:



Stabilizer

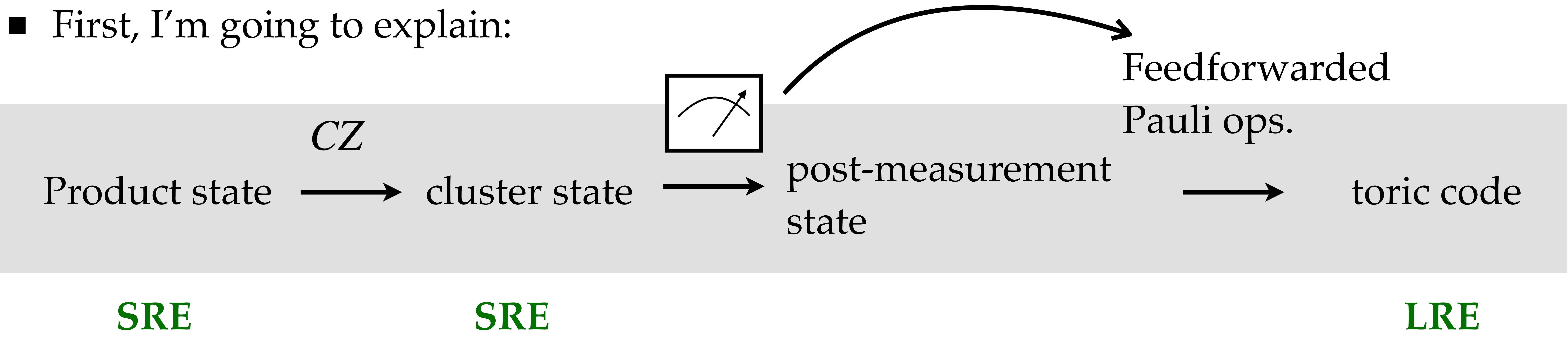


1-form symmetry

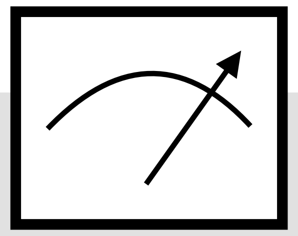
Measurement as a shortcut to topological orders

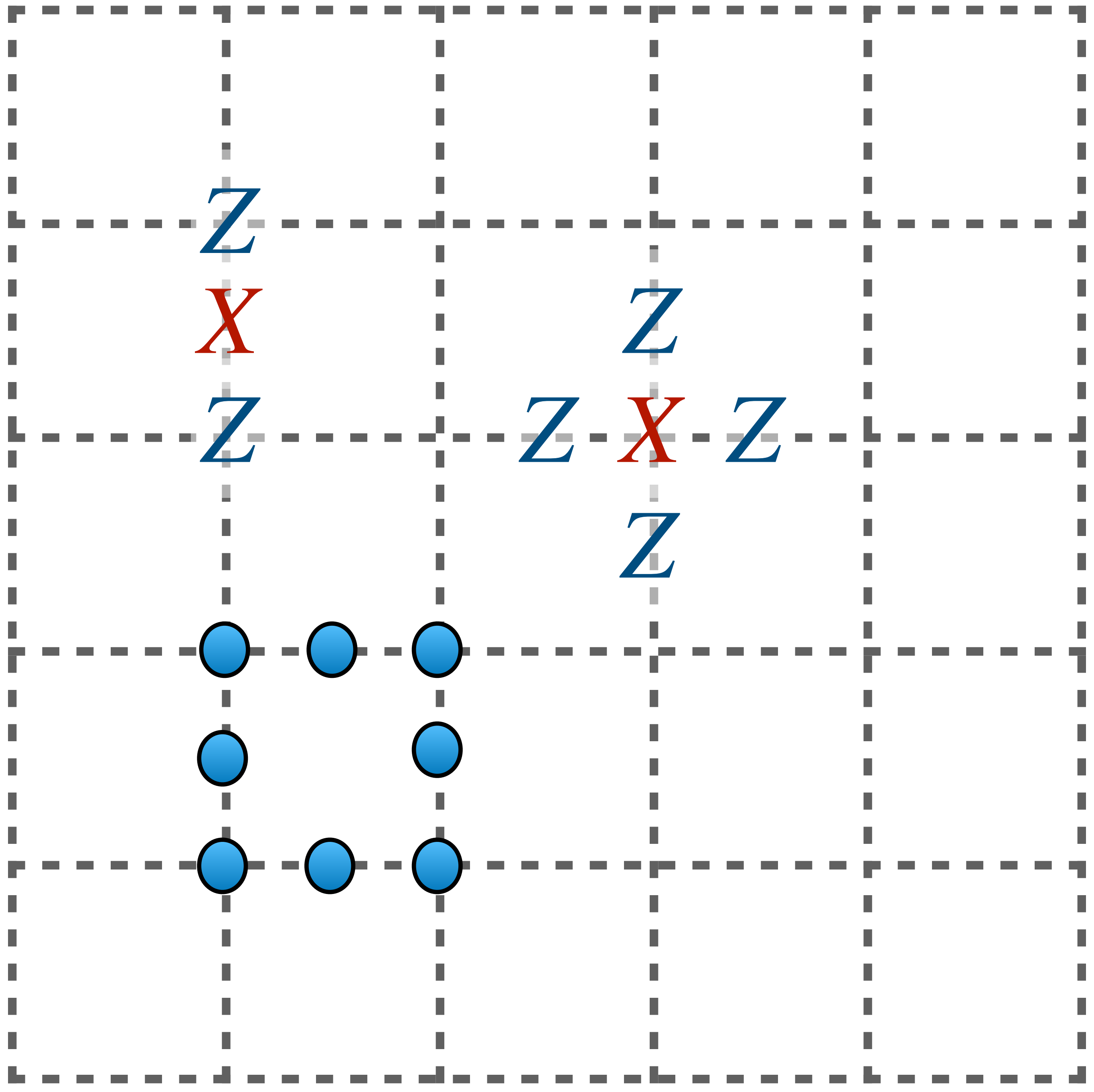
- The toric code cannot be prepared with finite-depth local unitaries from a product state.
- One obvious loophole is to use non-unitary operations. → Measurement ?

- Cluster-state (graph-state) entangler only produces short-range entanglement.
- This is because the CZ gates are mutually commutative. So one can apply the entangler at once, *i.e.*, the depth is 1.
- First, I'm going to explain:



Measurement as a shortcut to topological orders

Product state $\xrightarrow{CZ}$ cluster state ✔  $\longrightarrow$ post-measurement state $\xrightarrow{\text{Feedforwarded Pauli ops.}}$ toric code



- Cluster state on the Lieb lattice
- Qubits are placed on edges and vertices
- Apply CZ's to nearest-neighbor qubits.

* edge and vertex in the sense of the lattice, not a graph

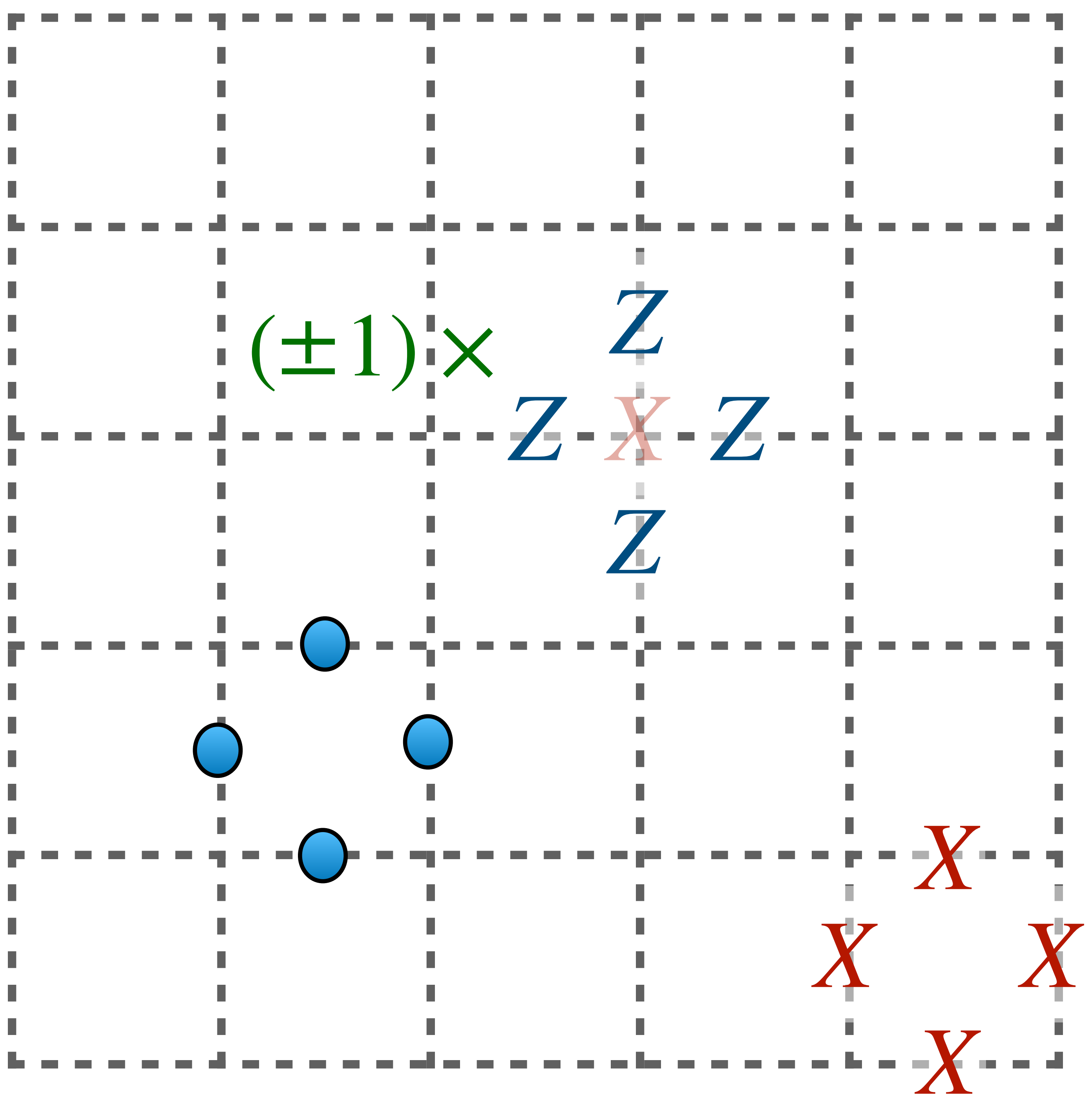
$$K_e = X_e \prod_{v \in e} Z_v\,,\quad K_v = X_v \prod_{e \ni v} Z_e$$

- There is a global symmetry in this cluster state.

$$\prod_v K_v |\psi_{\mathscr{C}}\rangle = \prod_v X_v |\psi_{\mathscr{C}}\rangle = |\psi_{\mathscr{C}}\rangle\,.$$

Measurement as a shortcut to topological orders

Product state $\xrightarrow{CZ}$ cluster state $\longrightarrow$ post-measurement state ✔ $\xrightarrow{\text{Feedforwarded Pauli ops.}}$ toric code



- Measure vertex qubits in the X basis.

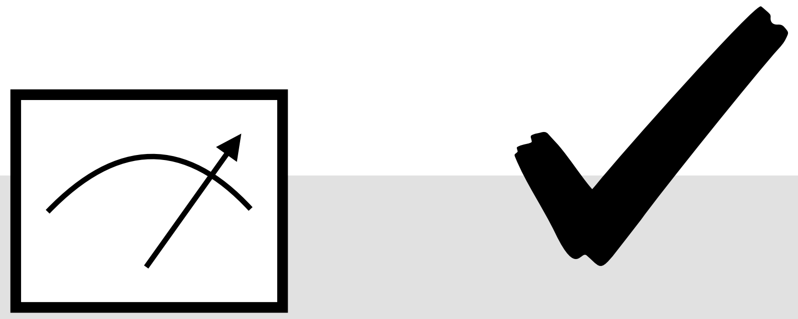
New stabilizers:

$$\pm X_v, \quad \pm \prod_{e \supset v} Z_e, \quad \prod_{e \subset p} X_e$$

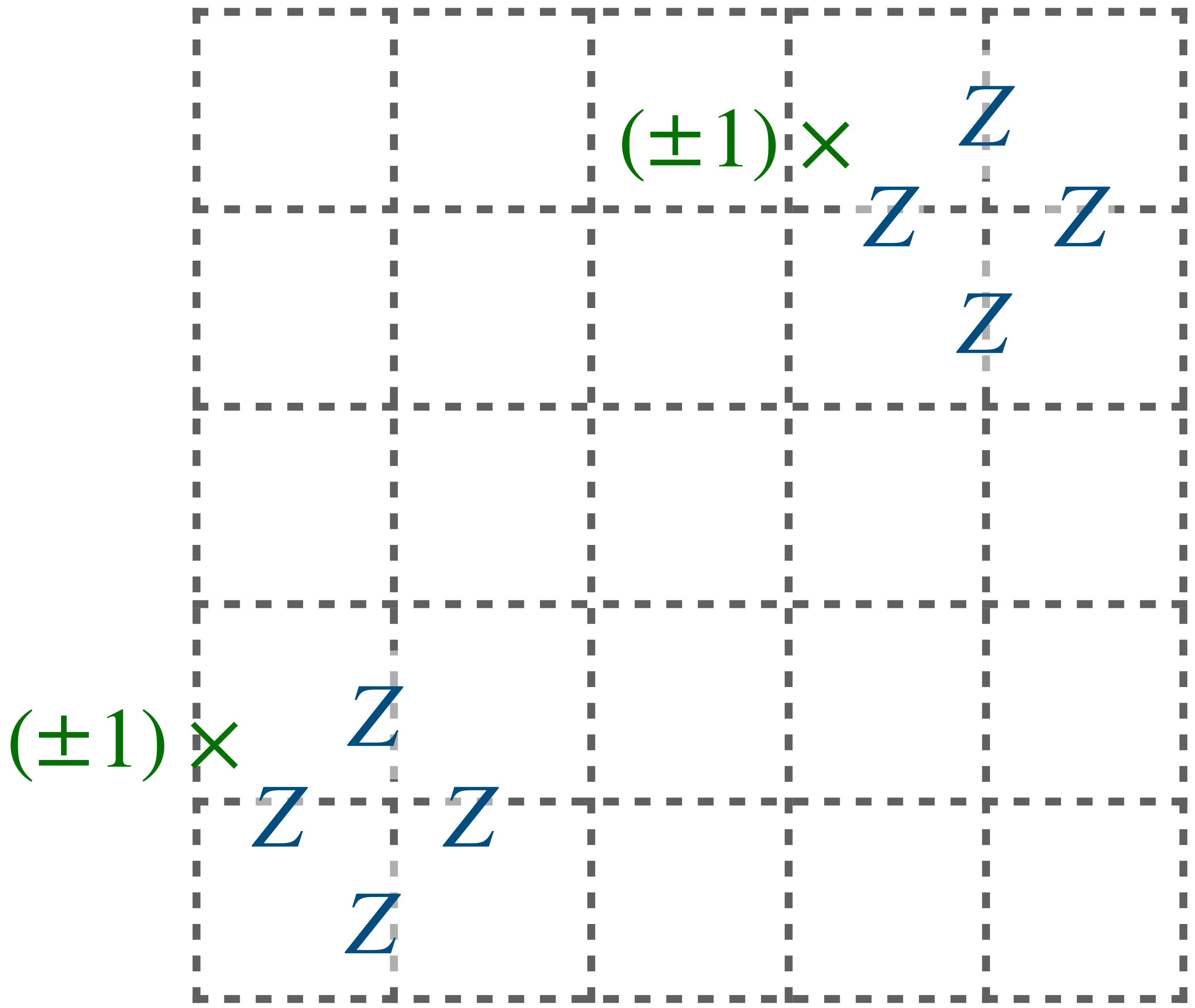
The last one is the product of K_e stabilizers around a plaquette p.
(K_e anti-commutes with X_v, but $\prod_{e \subset p} X_e$ commutes.)

It's not quite the ground state of the toric code…

Measurement as a shortcut to topological orders



Product state $\xrightarrow{CZ}$ cluster state $\longrightarrow$ post-measurement state $\xrightarrow{\text{Feedforwarded Pauli ops.}}$ toric code



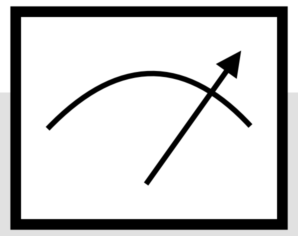
- The global symmetry constraints the measurement outcomes: $x_v = \pm 1$.

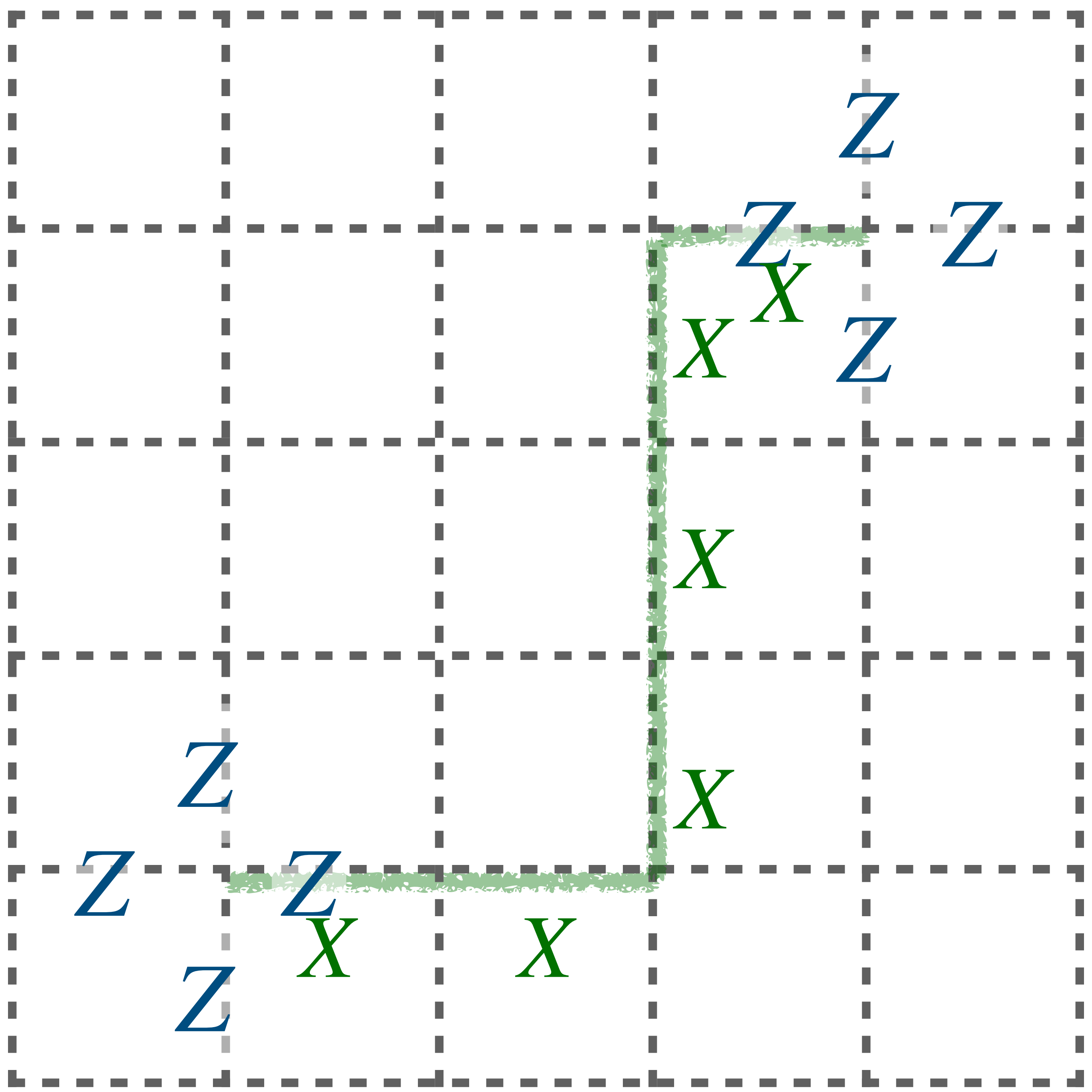
$$\prod_v x_v |\psi_{\mathscr{C}}\rangle = |\psi_{\mathscr{C}}\rangle .$$

This means that there are always *an even number of -1 outcomes*!

- This implies that the outcome state is the toric code ground state with string operators that pair up -1 outcomes. (Next slide)

Measurement as a shortcut to topological orders

Product state $\xrightarrow{CZ}$ cluster state  $\longrightarrow$ post-measurement state ✔ $\xrightarrow{\text{Feedforwarded Pauli ops.}}$ toric code



- Left figure:

The outcome state can be written as

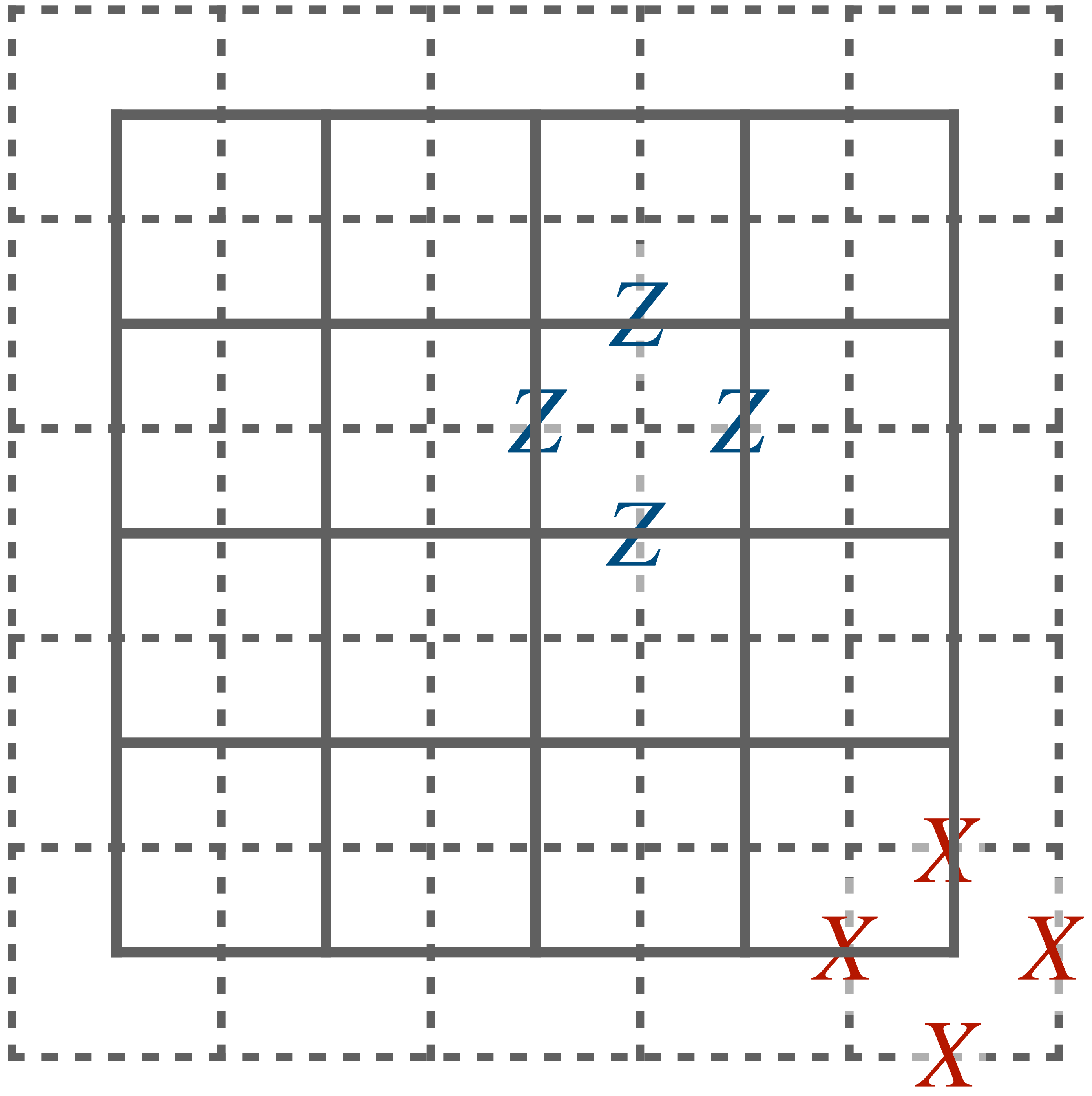
$$\left(\prod_{e\in \text{string}} X_e\right)|gs\rangle$$

Indeed, at the endpoints of the string, Z stabilizers are flipped.

The shape of the path doesn't matter, as the X stabilizer can deform strings.

Measurement as a shortcut to topological orders

Product state $\xrightarrow{CZ}$ cluster state $\longrightarrow$ post-measurement state $\xrightarrow{\text{Feedforwarded Pauli ops.}}$ toric code ✔



- One can counter the randomness by applying Pauli X operators.

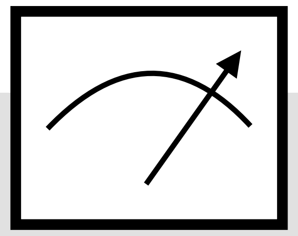
$$\left(\prod_{e\in \text{strings}} X_e\right)|\text{out}\rangle = |gs\rangle$$

- *Fin.*

Measurement as a shortcut to topological orders

The technique can be generalized for any $\mathbb{Z}_2$ (and some other discrete groups) symmetric state. [Tantivasadakarn-Thorngren-Vishwanath-Verresen (2021)] [Lu-Lessa-Kim-Hsieh (2022)] etc.

$|\Psi_{\text{sym}}\rangle_V \otimes |+\rangle^{\otimes E} \xrightarrow{CZ}$ SPT



$\longrightarrow$ post-measurement state $\xrightarrow{\text{Feedforwarded Pauli ops.}}$ Topological order

The operations in total yields measurement-based *Kramers-Wannier-Wegner transformation*

$$\mathsf{KW} = \langle +|^V \prod CZ_{e,v} |+\rangle^E$$

As we'll see, the toric code is an example and a special limit of lattice gauge theories.

$$H_{\text{gauge theory}} \, \mathsf{KW} = \mathsf{KW} \, H_{\text{Ising}}$$

KW can be seen as a space-like interface between two dual theories.

Measurement as a shortcut to topological orders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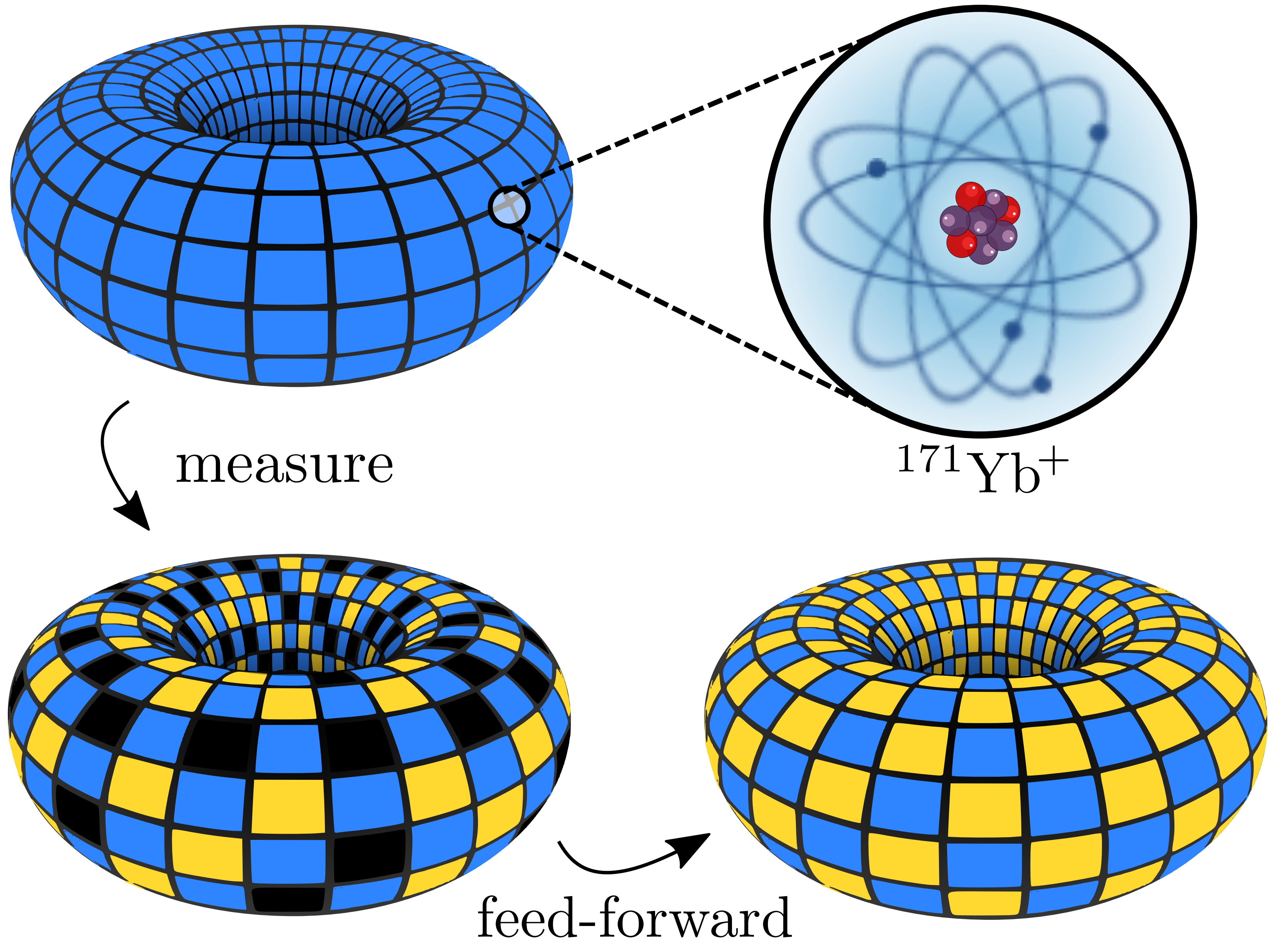
NEWS | 09 May 2023

Physicists create long-sought topological quantum states

Exotic particles called nonabelions could fix quantum computers' error problem.

Davide Castelvecchi



M. Iqbal et al. arXiv:2305.03766

M. Iqbal et al. arXiv:2302.01917

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Hamiltonian lattice gauge theories

Let us start with (2+1)d transverse-field Ising model, which is equivalent to the 3d classical Ising model. I explain the connection between the two. Cf. [J. Kogut (1976)]

$$Z_{\text{Ising}} = \sum_{\{s_v = \pm 1\}} e^{-\beta I[s]}$$

where

$$I[s] = -K \sum_{e} \prod_{v \subset e} s_v \,.$$

is the Ising Hamiltonian on the 3d square lattice.

We take one direction, say the z direction, as a special direction and make the coupling constant anisotropic.

$$I_{\text{anis.}}[s] = -K_s \sum_{e \in E_x \cup E_y} \prod_{v \subset e} s_v - K_t \sum_{e \in E_z} \prod_{v \subset e} s_v$$

We view the x and y directions as spatial, and z as temporal.

Hamiltonian lattice gauge theories

A simple rewriting gives us

$$I_{\text{anis.}}[s] = -K_s \sum_{e\in E_x\cup E_y} \prod_{v\subset e} s_v - K_t \sum_{e\in E_z} \prod_{v\subset e} s_v$$

$$\sim -K_s \sum_{e\in E_x\cup E_y} \prod_{v\subset e} s_v + \frac{K_t}{2} \sum_{e\in E_z} (s_{v(e)_+} - s_{v(e)_-})^2$$

up to a constant. Here,

$$v(e)_+ = \{x, y, z+1\} \text{ and } v(e)_- = \{x, y, z\} \text{ for } e = \{x, y\} \times [z, z+1]\,.$$

To derive a 2d quantum Hamiltonian related via

$$Z_{\text{Ising}} \simeq \mathrm{Tr}\left(e^{-\tau H}\right)$$

we take the spin variable as the basis of the Hilbert space. We also take an approximation $e^{-\tau H} \simeq (e^{-\Delta\tau H})^N$.

At each temporal slice $z = \text{int.}$, we insert a complete basis $\bigotimes_{v\in V_{z=j}} |s_v\rangle\langle s_v|$

Hamiltonian lattice gauge theories

We aim to find H such that

$$Z_{\text{Ising}} \simeq \mathrm{Tr}\Big(\bigotimes_{v\in V_j}\langle s_v| e^{-\Delta\tau H}\bigotimes_{v'\in V_{j+1}}|s_{v'}\rangle\Big)^N .$$

Relate parameters as

$$\beta K_s = \lambda e^{-2\beta K_t},\ \ \Delta\tau = e^{-2\beta K_t},\ \ \beta K_t \to \infty\ \ (\text{small } \Delta\tau \text{ limit}) .$$

First look at the diagonal transfer matrix elements:

$$\exp\Big(-\beta K_s \sum_{e\in E_x\cup E_y}\prod_{v\subset e} s_v\Big) \longleftrightarrow \exp\Big(-\Delta\tau \sum_{e\in E_x\cup E_y}\prod_{v\subset e} Z_v\Big) \text{ for each } z \text{ slice.}$$

So we have

$$H_{\text{diag}} = -\lambda\sum_{e\in E}\prod_{v\subset e} Z_v .$$

Hamiltonian lattice gauge theories

We aim to find H such that

$$Z_{\text{Ising}} \simeq \text{Tr}\Big(\bigotimes_{v\in V_j} \langle s_v | e^{-\Delta\tau H} \bigotimes_{v'\in V_{j+1}} | s_{v'} \rangle\Big)^N .$$

Relate parameters as

$$\beta K_s = \lambda e^{-2\beta K_t},\ \ \Delta\tau = e^{-2\beta K_t},\ \ \beta K_t \to \infty\ \ (\text{small } \Delta\tau \text{ limit}) .$$

Next look at a single-shift transition. Say $\{s_v\}$ and $\{s_{v'}\}$ differ at one site between j and $j+1$. Due to the term $-\beta\frac{K_t}{2}\sum_{e\in E_z}(s_{v(e)_+} - s_{v(e)_-})^2$, the Boltzmann factor gains a weight $e^{-2\beta K_t}$.

We identify as

$$\langle \{s_v\} | (-\Delta\tau H) | \{s_{v'}\} \rangle \simeq e^{-2\beta K_t} \equiv \Delta\tau.$$

This is generated by

$$H_{\text{off-diag}} = -\sum_{u\in V} X_u .$$

Hamiltonian lattice gauge theories

In total, we have for 3d classical Ising model (in a certain limit) that

$$Z_{\text{Ising}} \simeq \text{Tr}(e^{-\Delta\tau H})^N$$

with

$$H = H_{\text{TFI}} = -\sum_{v\in V} X_v - \lambda \sum_{e\in E} \prod_{v\subset e} Z_v$$

where the vertices and edges are those in 2-dimensions (xy-slices).

This construction straightforwardly generalizes to classical Ising models in arbitrary dimensions and we get (quantum) transverse-field Ising models in one-dimension lower.

This also generalizes to lattice gauge theories. (Next slide)

Hamiltonian lattice gauge theories

Consider the $G = \mathbb{Z}_2$ version of Wilson's plaquette action:

$$I[\{u_e = \pm 1\}] = -J \sum_{p \in P} \prod_{e \subset p} u_e \,.$$

The action in invariant under the simultaneous flip of spins on edges (links) around a vertex.

We again make the coupling constants anisotropic.

We make use of the gauge transformation to fix spins on temporal edges (temporal link variables) to 1. Then we get

$$I[\{u_e = \pm 1\}] = -J_s \sum_{p \in P_{xy}} \prod_{e \subset p} u_e - J_t \sum_{p \in P_{\bullet z}} u_{e(p)_+} u_{e(p)_-}$$

where $e(p)_+$ and $e(p)_-$ are edges in the plaquette p at larger and smaller 'temporal' coordinate, respectively.

Just as in the study with Ising models, we can again use $u_{e(p)_+} u_{e(p)_-} = -\frac{1}{2}(u_{e(p)_+} - u_{e(p)_-})^2 + 1$

Hamiltonian lattice gauge theories

We have for d-dim Euclidean path integral of the lattice gauge theory that

$$Z_{\text{Gauge}} \simeq \mathrm{Tr}(e^{-\Delta\tau H})^N$$

with

$$H = H_{\text{Gauge}} = -\sum_{e\in E} X_e - \lambda \sum_{p\in P}\prod_{e\subset p} Z_e$$

where the edges and plaquettes are those in $(d-1)$-dimensions.

We already used the gauge redundancy to fix the temporal link variables to 1. However, there is residual gauge redundancy, which is generated by simultaneous gauge transformations over temporal coordinates at a fixed vertex in the spatial slice.

In terms of the quantum system, this is generated by the Gauss law divergence operator

$$G_v = \prod_{e\supset v} X_e\,.$$

One can check that $[H_{\text{Gauge}}, G_v] = 0\,.$

Hamiltonian lattice gauge theories

- Toric code:

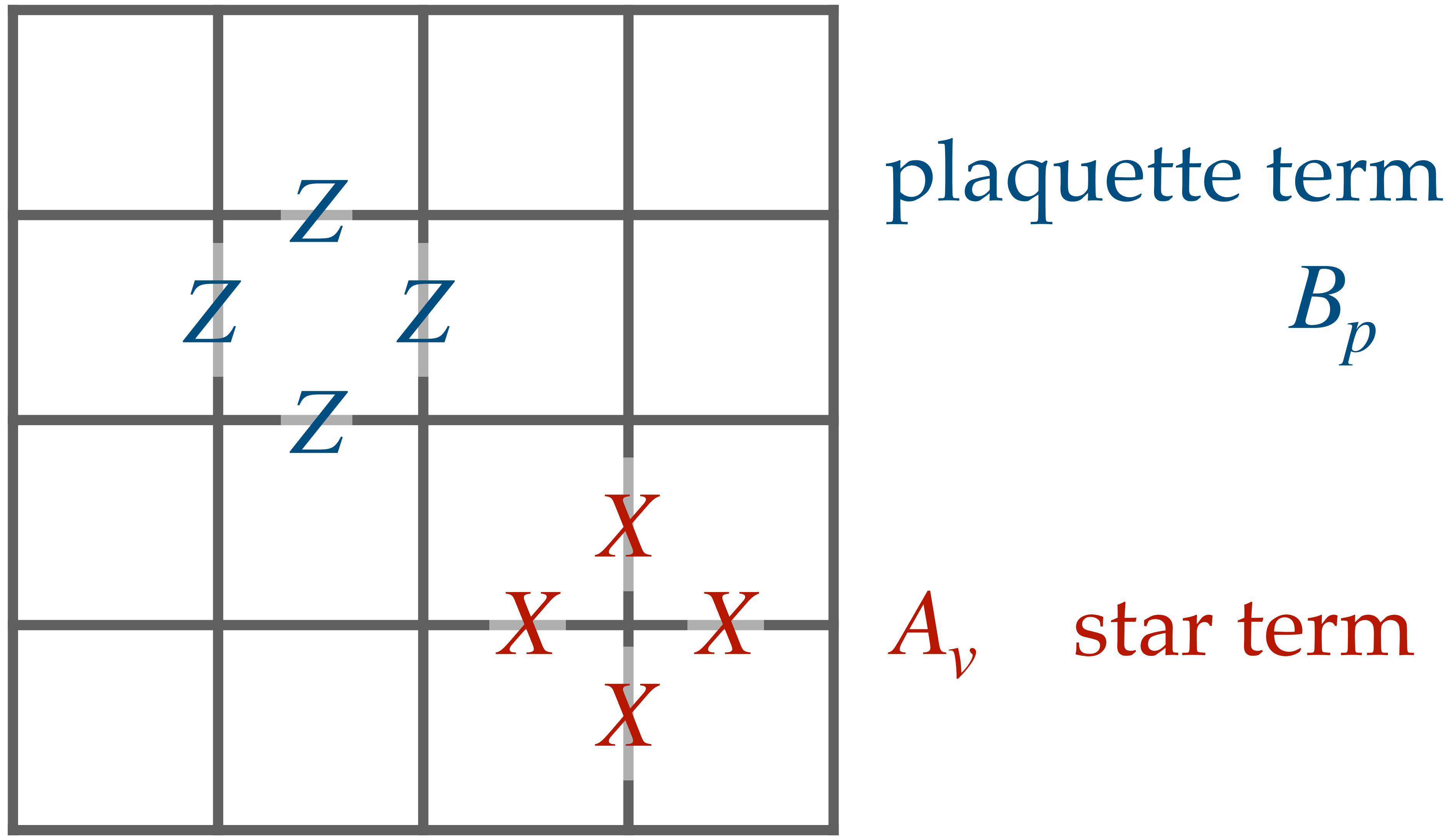
$$H_{\mathrm{TC}} = -\sum_v A_v - \sum_p B_p$$

- The $\mathbb{Z}_2$ lattice gauge theory may be written as

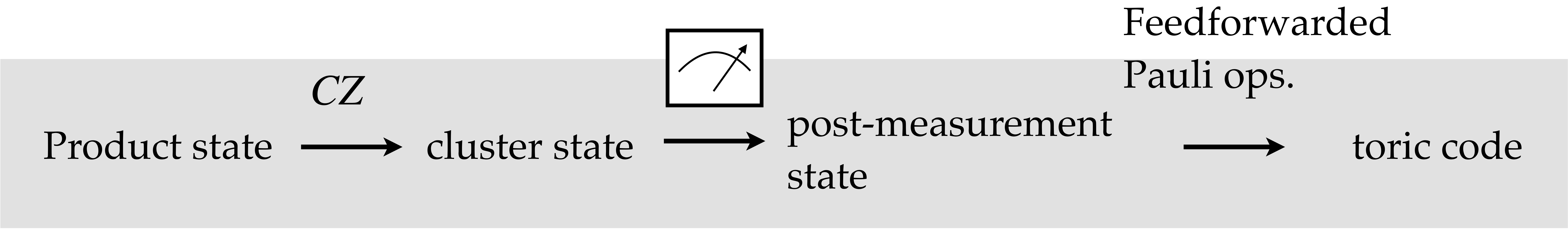
$$H_{\mathrm{Gauge}} = -\sum_{e\in E} X_e - \lambda \sum_{p\in P} B_p$$

with $G_v = A_v = 1$.

- In condensed matter physics, the toric code (with some extra terms) is often referred to as a 'lattice gauge theory' in this sense.

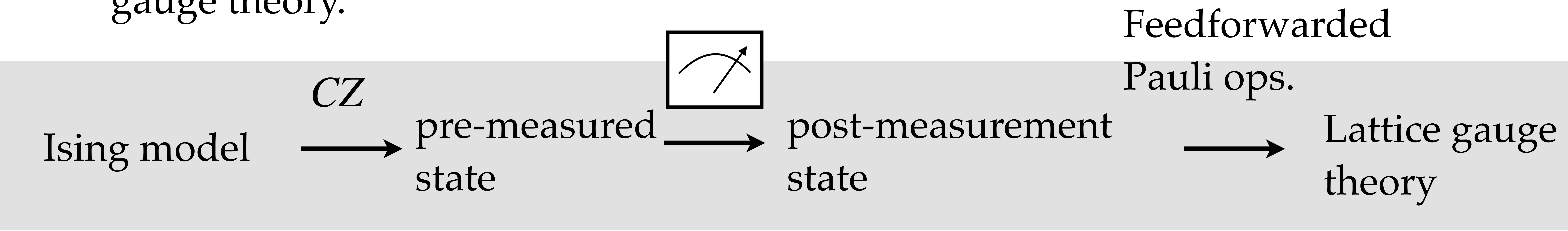


Hamiltonian lattice gauge theories

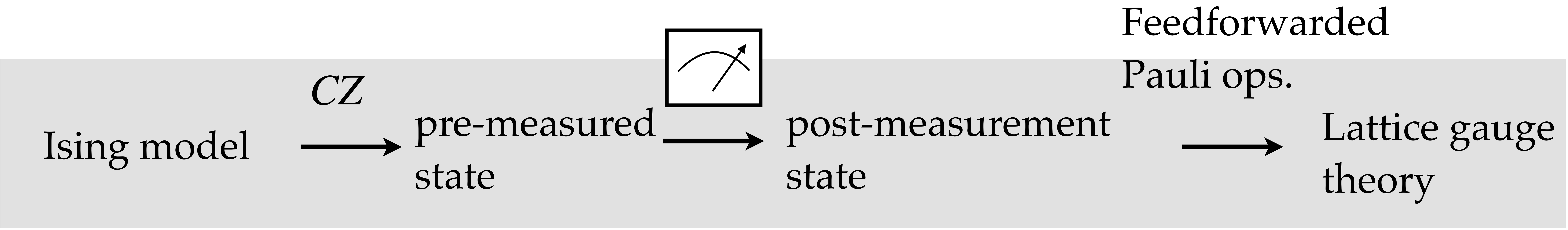


We ask, is there a generalization of the measurement-based preparation of the toric code to that of lattice gauge theories?

It turns out that the method above can indeed implement the Kramers-Wannier-Wegner duality transformation from the Ising model to the lattice gauge theory.

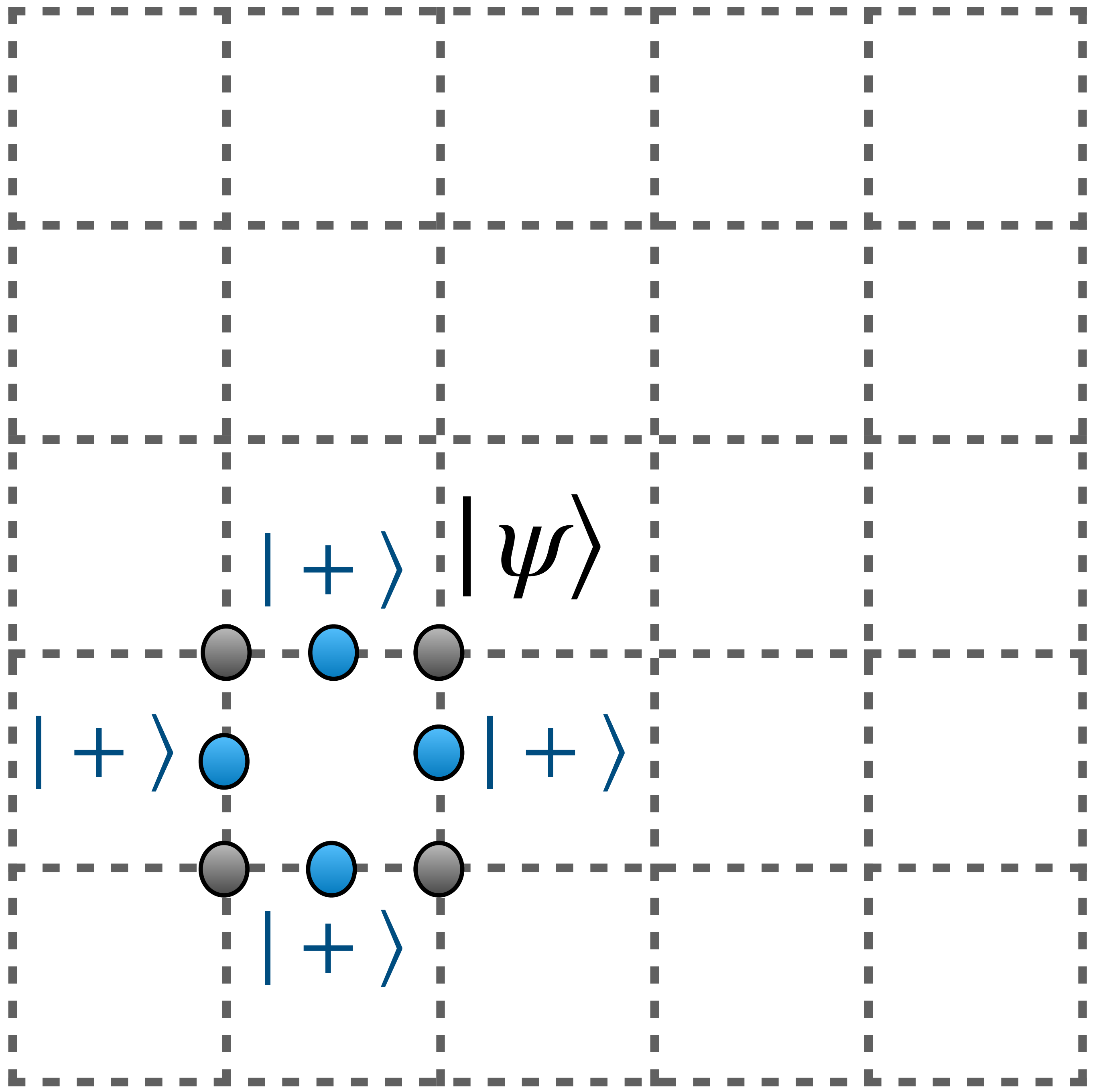


Hamiltonian lattice gauge theories



- Start with a state on vertices $|\psi\rangle$
- Introduce ancilla d.o.f. on edges $|+\rangle^{\otimes E}$
- Apply the cluster-state entangler $\mathcal{U}_{CZ} = \prod_{e \in E} \prod_{v \subset e} CZ_{e,v}$
- Measure vertex d.o.f. in the X basis
- As described previously, perform corrections against randomness. This is possible if we have an even number of $|-\rangle$ outcomes. (Post-select.)
- All put together, we are implementing an operator

$$\mathsf{KW} = \langle + |^{\otimes V} \mathcal{U}_{CZ} | + \rangle^{\otimes E} \qquad \mathsf{KW} : \mathcal{H}_V \to \mathcal{H}_E$$



Hamiltonian lattice gauge theories

$\mathsf{KW} = \langle + |^{\otimes V} \mathcal{U}_{CZ} | + \rangle^{\otimes E}$ with $\mathcal{U}_{CZ} = \prod_{e \in E} \prod_{v \subset e} CZ_{e,v}$ implements the following map:

$$X_e \, \mathsf{KW} = \mathsf{KW} \, Z_{v(e)_1} Z_{v(e)_2}$$

$$Z_{e(v)_1} Z_{e(v)_2} Z_{e(v)_3} Z_{e(v)_4} \, \mathsf{KW} = \mathsf{KW} \, X_v$$

In the dual lattice picture, $X_e = X_{e*}$ and $Z_{e(v)_1} Z_{e(v)_2} Z_{e(v)_3} Z_{e(v)_4} = Z_{e*(p*)_1} Z_{e*(p*)_2} Z_{e*(p*)_3} Z_{e*(p*)_4} = B_{p*}$.

$$\mathsf{KW} \cdot H_{\text{Ising}} = H_{\text{Gauge}} \mathsf{KW}$$

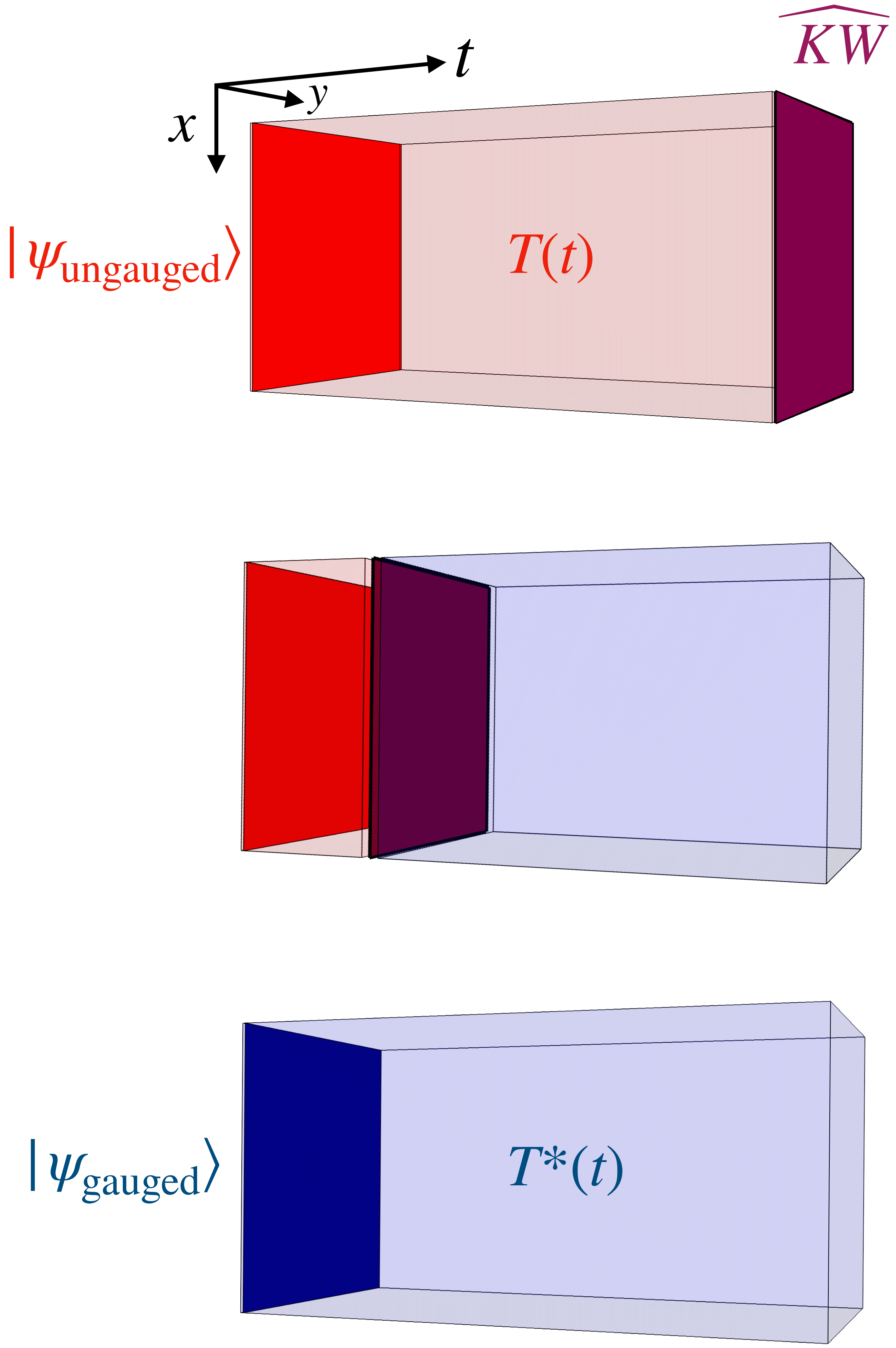
This is a *gauging* operation such that

$$\mathsf{KW} \cdot \prod_{v \in V} X_v = \mathsf{KW} \quad \text{(global symmetry in } \mathcal{H}_V \text{ gets trivialized)}$$

$$\mathsf{KW} = G_{v*} \cdot \mathsf{KW} \quad \text{(Gauss law in } \mathcal{H}_E \text{ emerges)}$$

Hamiltonian lattice gauge theories

This may be used for a quantum simulation. Suppose we start with a state that satisfies $\prod_{v\in V} X_v |\psi\rangle = |\psi\rangle$ (to ensure that the number of the $|-\rangle$ outcome is even).



A real-time evolution

$$e^{-itH_{\text{Ising}}}|\psi\rangle$$

can be transformed by the measurement-based gauging procedure as

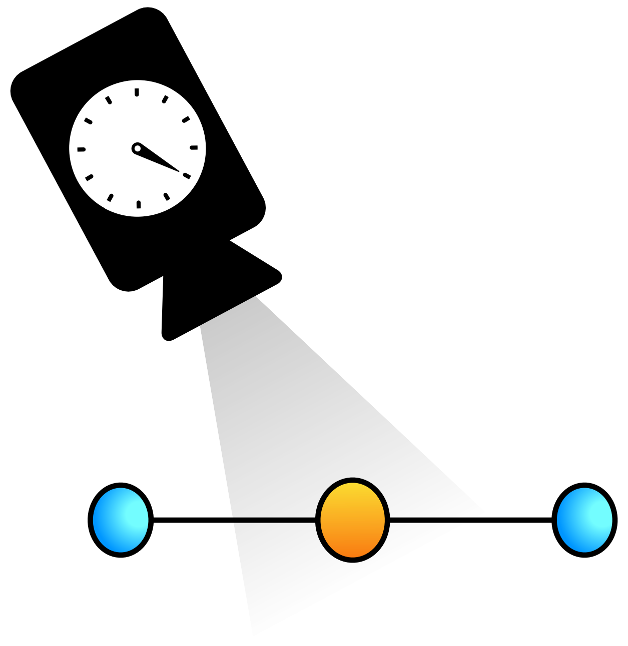
$$\mathsf{KW}e^{-itH_{\text{Ising}}}|\psi\rangle = e^{-itH_{\text{Gauge}}}\mathsf{KW}|\psi\rangle\ .$$

When the state $|\psi\rangle$ is in the paramagnetic phase ($\simeq |+\rangle^{\otimes V}$), then the gauged state $\mathsf{KW}|\psi\rangle$ is in the deconfining phase ($\simeq$ toric code).

Hamiltonian lattice gauge theories

- By a Lieb-Robinson bound [Bravyi-Hastings-Verstraete], it is expected that a state in the toric code phase cannot be obtained by a constant-depth unitary circuit. Measurement supplies non-unitarity to give a short-cut to a quantum simulation in the deconfining regime. [Ashkenazi-Zohar (2021), HS-Wei (2023)]
- The idea of performing KW on the Ising quantum simulation could be implemented on real quantum devices in the near future, as the Ising quantum simulation requires less connectivity.
- In (3+1)dimensions, the lattice $\mathbb{Z}_2$ gauge theory is self-dual. Gauging may not be so useful as a short cut for simulating such models.
- Below, we consider a quantum simulation scheme motivated by MBQC.

A formula



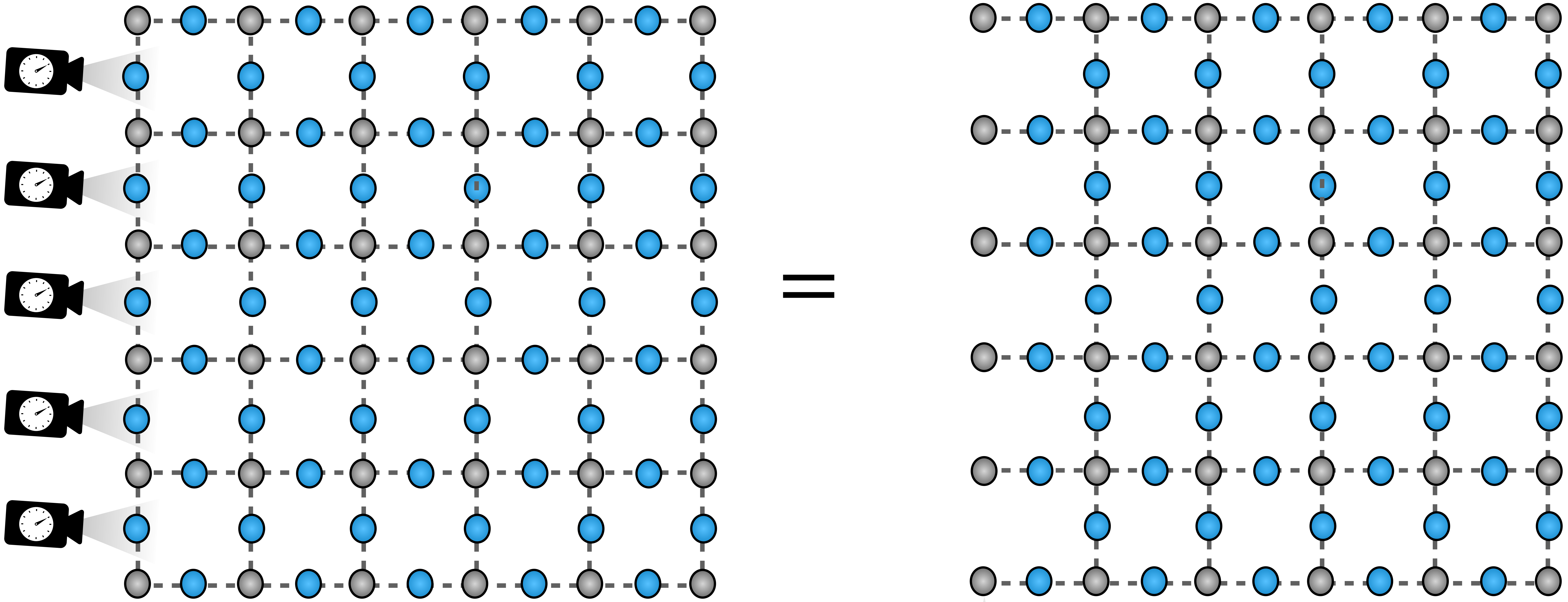
- Consider a general "initial state" $|\psi\rangle_{bc}$
- Prepare a "resource state" $CZ_{a,b}CZ_{a,c}|\psi\rangle_{bc}|+\rangle_a$
- Measure the middle qubit with $\{e^{i\xi X}|0\rangle, e^{i\xi X}|1\rangle\}$, i.e., $X^s e^{i\xi X}|0\rangle \quad (s=0,1)$

$$\langle 0|_a e^{-i\xi X_a} X_a^s \cdot CZ_{a,b}CZ_{a,c}|\psi\rangle_{bc}|+\rangle_a = e^{-i\xi Z_b Z_c}(Z_b Z_c)^s|\psi\rangle_{bc}$$

$\rightarrow$ **Multi-qubit rotation.**

Cluster state for quantum simulation

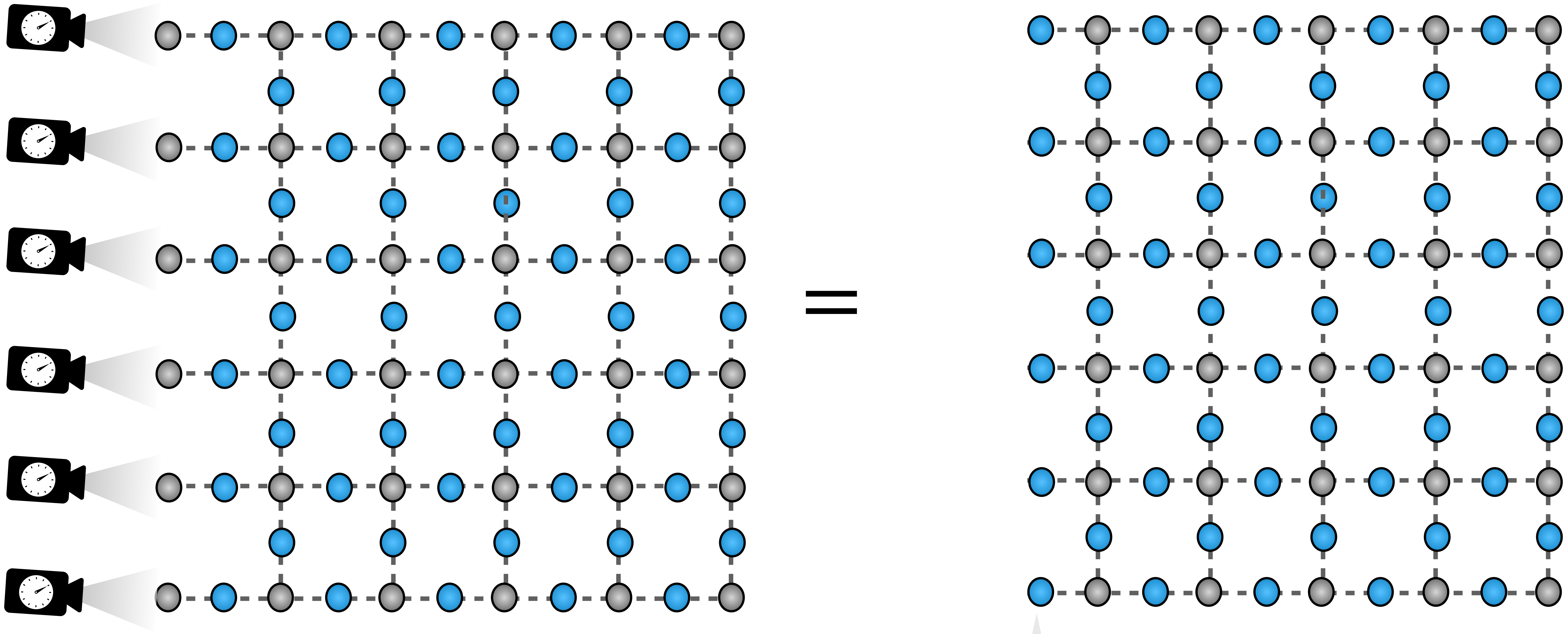
- Simulating (1+1)d transverse-field Ising model on the 2d cluster state



$$\prod_e \left(Z_{v(e)_+} Z_{v(e)_-}\right)^{s(e)} e^{-i\xi Z_{v(e)_+} Z_{v(e)_-}}$$

Cluster state for quantum simulation

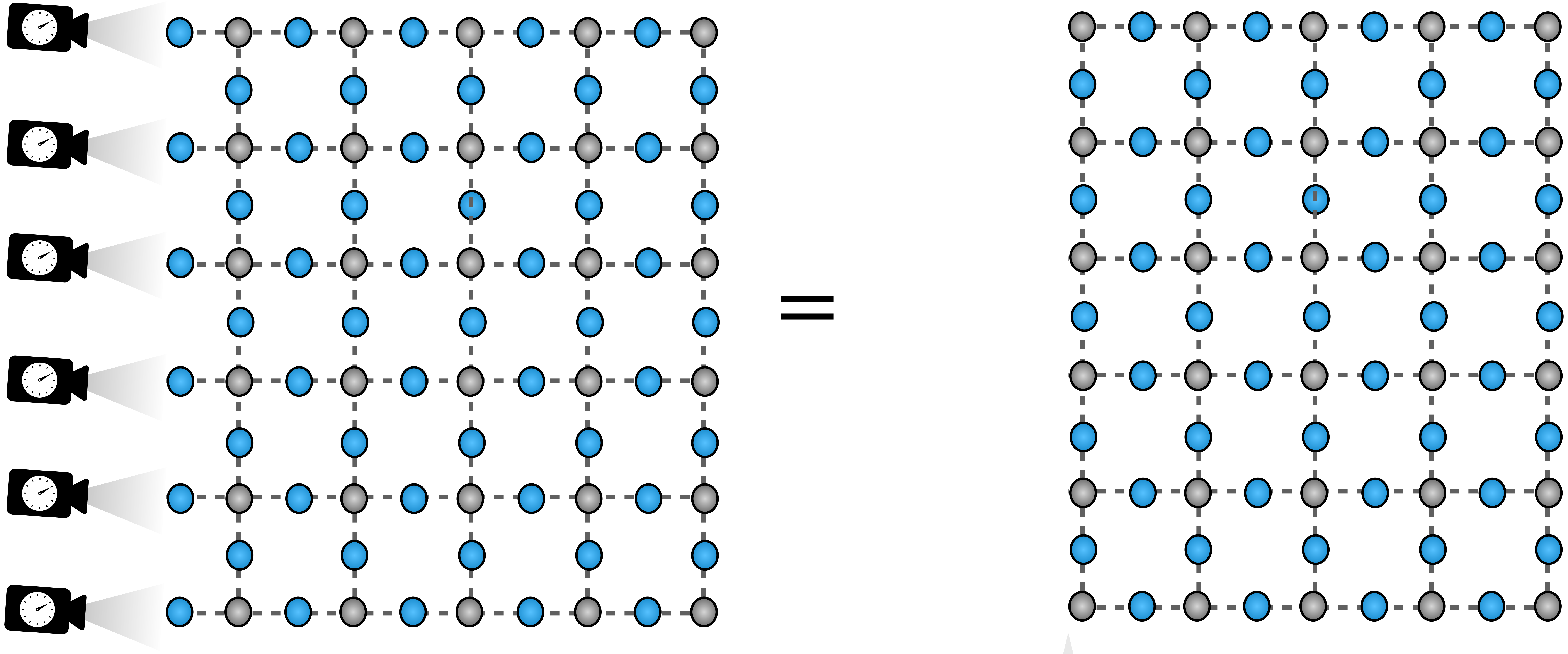
- Simulating (1+1)d transverse-field Ising model on the 2d cluster state



$$\prod_v H_v (Z_v)^{s(v)}$$

Cluster state for quantum simulation

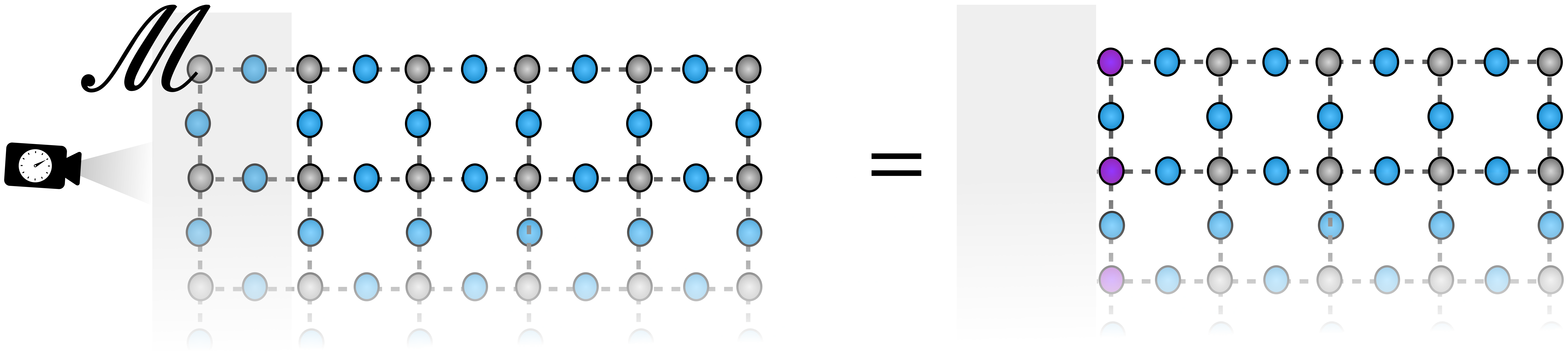
- Simulating (1+1)d transverse-field Ising model on the 2d cluster state



$$\prod_{v} H_v (Z_v)^{s(v)} e^{-i\xi Z_v}$$

Cluster state for quantum simulation

- Simulating (1+1)d transverse-field Ising model on the 2d cluster state



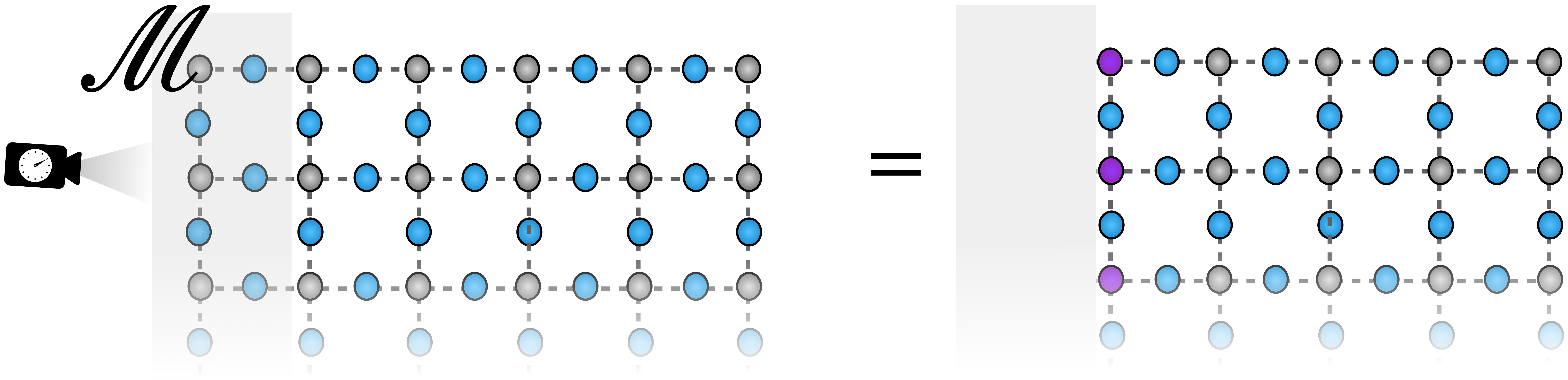
$$\mathscr{M} \cdot \left[\mathscr{U}_{CZ} |\phi\rangle^{(x=0)}_{\text{edge}} \otimes |+\rangle_{\text{others}}\right]$$

$$= \mathscr{U}_{CZ}\left(\mathscr{O}_{\text{bp}} \cdot \prod_v H_v e^{-i\xi Z_v} H_v \prod_e e^{-i\xi' Z_{v(e)_+} Z_{v(e)_-}} |\phi\rangle^{(x=1)}_{\text{edge}} \otimes |+\rangle_{\text{others}}\right)$$

$$= \mathscr{U}_{CZ}\left(\mathscr{O}_{\text{bp}} \cdot \prod_v e^{-i\xi X_v} \prod_e e^{-i\xi' Z_{v(e)_+} Z_{v(e)_-}} |\phi\rangle^{(x=1)}_{\text{edge}} \otimes |+\rangle_{\text{others}}\right)$$

Cluster state for quantum simulation

- Simulating (1+1)d transverse-field Ising model on the 2d cluster state



$$\mathscr{M} \cdot \left[\mathscr{U}_{CZ} | \phi \rangle^{(x=0)}_{\text{edge}} \otimes | + \rangle_{\text{others}} \right]$$

$$= \mathscr{U}_{CZ} \left(\mathscr{O}_{\text{bp}} \cdot U_{\text{TFI}}(\Delta t) | \phi \rangle^{(x=1)}_{\text{edge}} \otimes | + \rangle_{\text{others}} \right)$$

Plan

Part I: Quantum computation by measurement

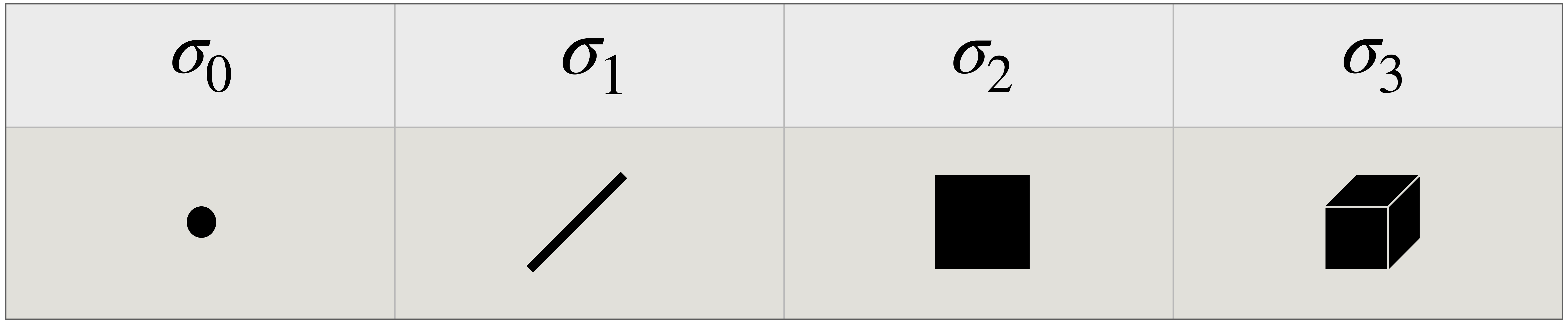
- Stabilizer formalism
- Graph state
- Gate teleportation
- Universal quantum computation on graph states

Part II: Measurement-based quantum computation and lattice gauge theory

- Measurement as a shortcut to topological orders
- $\mathbb{Z}_2$ lattice gauge theory
- Quantum simulation of lattice gauge theories

Wegner’s generalized Ising models

Cell simplex σ_i



$\breve{\sigma}_i$: cell simplices in d dimensional hypercube lattice

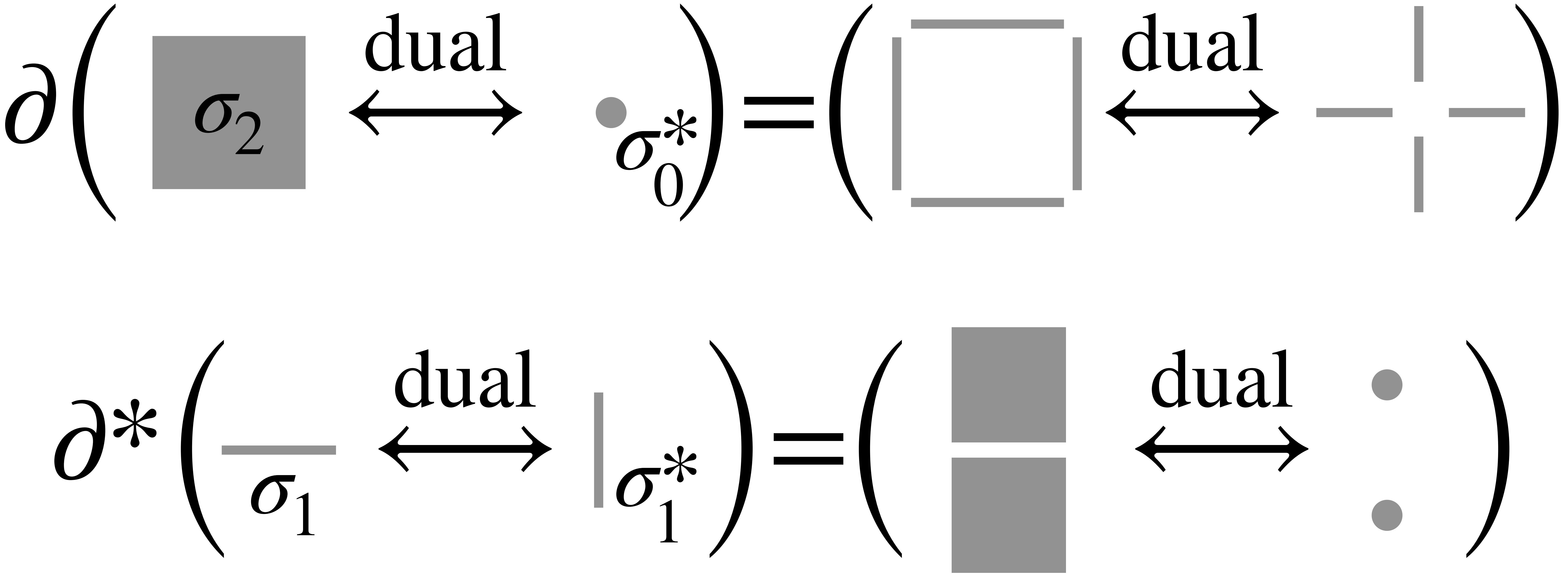
σ_i : cell simplices in $d-1$ dimensional hypercube lattice

$$\breve{\sigma}_i = \sigma_i \times \{j\} \text{ or } \breve{\sigma}_{i+1} = \sigma_i \times [j, j+1]$$

Point Interval x_d coordinate

Similarly, we have cell simplices in the dual lattice with $\sigma_i \simeq \sigma^*_{d-i}$.

We have $\partial^2 = 0$ (and $(\partial^*)^2 = 0$) and a chain complex.



Wegner's generalized Ising model

Model $M_{(d,n)}$:
Classical spin variables $S_{\breve{\sigma}_{n-1}} \in \{+1, -1\}$ living on $(n-1)$-cells in the d-dimensional hybercubic lattice. [Wegner (1971)]

Euclidean action (classical Hamiltonian) I :

$$I = -J \sum_{\breve{\sigma}_n} \Big(\prod_{\breve{\sigma}_{n-1} \subset \partial \breve{\sigma}_n} S_{\breve{\sigma}_{n-1}} \Big) .$$

Via the transfer matrix formalism, we obtain a quantum Hamiltonian in $(d-1)$ dimensions with the continuous time.

$$H_{(d,n)} = -\sum_{\sigma_{n-1}} X(\sigma_{n-1}) - \lambda \sum_{\sigma_n} Z(\partial \sigma_n) .$$

Wegner's generalized Ising model

$M_{(d,1)}$

Classical Ising model

$$I = -J \sum_{\text{edge}} S(\partial \breve{\sigma}_1)$$

site variable

Transverse field Ising model

$$H_{(d,1)} = -\sum_{\sigma_0} X(\sigma_0) - \lambda \sum_{\sigma_1} Z(\partial \sigma_1)$$

$M_{(d,2)}$

Gauge theory (Wilson's plaquette action for $G = \mathbb{Z}_2$)

$$I = -J \sum_{\text{plaquette}} S(\partial \breve{\sigma}_2)$$

link variable

Quantum pure gauge theory

$$H_{(d,2)} = -\sum_{\sigma_1} X(\sigma_1) - \lambda \sum_{\sigma_2} Z(\partial \sigma_2)$$

Wegner's generalized Ising model

We wish to simulate a Trotterized (real) time evolution:

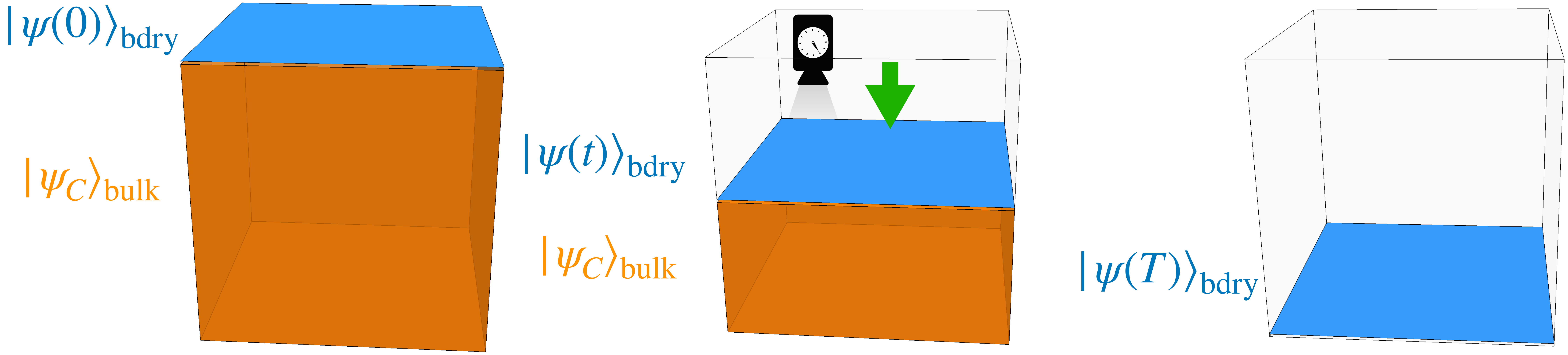
$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

with

$$T(t = j\Delta t) = \left(\prod_{\sigma_{n-1}} e^{i\Delta t X(\sigma_{n-1})} \prod_{\sigma_n} e^{i\Delta t \lambda Z(\partial\sigma_n)}\right)^j .$$

MBQS of lattice gauge theories

MBQS



$|\psi(t)\rangle_{\text{bdry}}$: **simulated state of $M_{(d,n)}$** with the Trotterized time evolution $T(t)$,

$$|\psi(t)\rangle_{\text{bdry}} = T(t)\,|\psi(0)\rangle\ .$$

$|\psi_C\rangle_{\text{bulk}}$: **resource state** to be measured — **generalized cluster state (gCS)**.

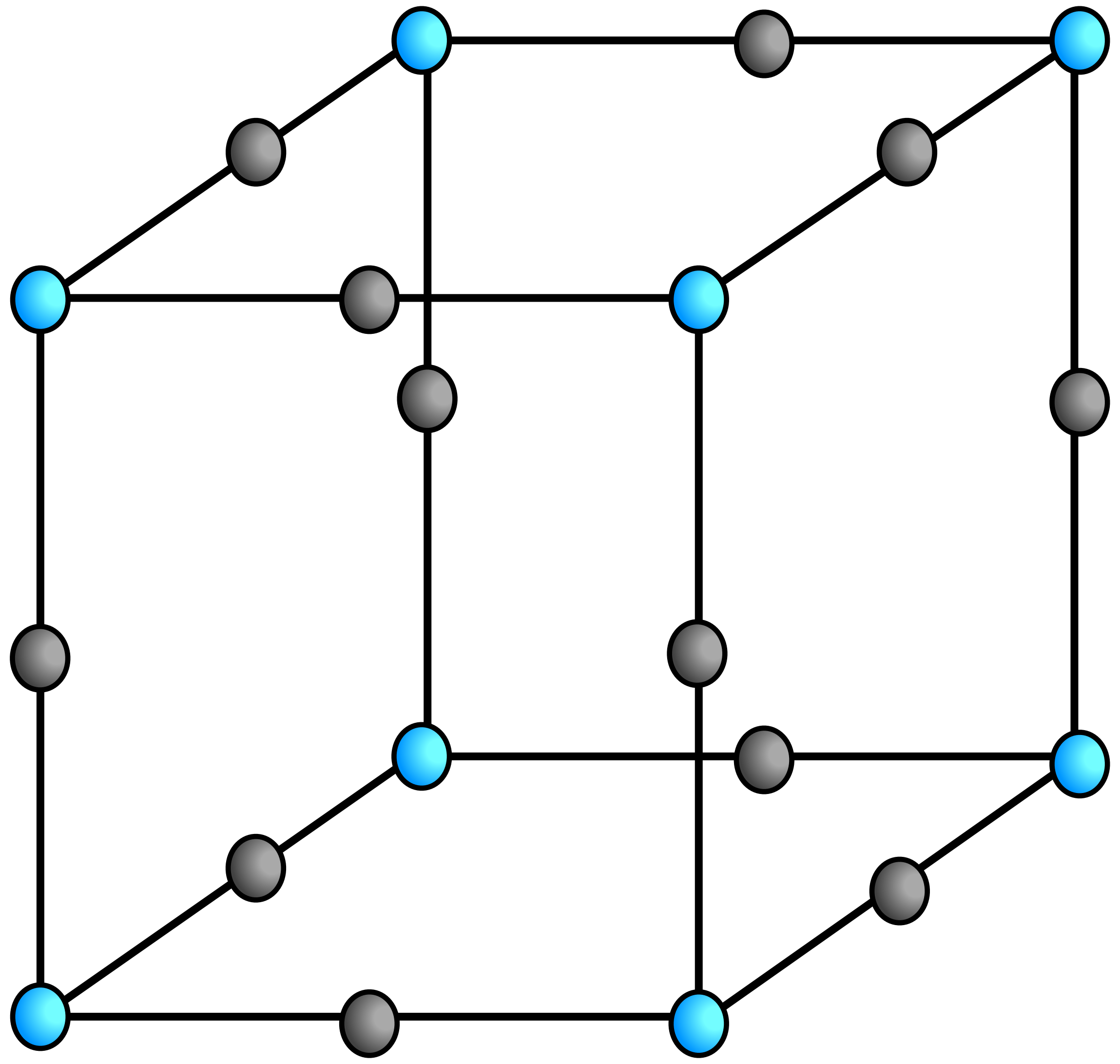
MBQS

Entanglement in our resource state, $gCS_{(d,n)}$ (generalized cluster state), is **tailored** to reflect the space-time structure of the model $M_{(d,n)}$:

$$|gCS_{(d,n)}\rangle := \mathcal{U}_{CZ}|+\rangle^{\breve{\Delta}_n}|+\rangle^{\breve{\Delta}_{n-1}}$$

$$\mathcal{U}_{CZ} = \prod_{\breve{\sigma}_n \in \breve{\Delta}_n} \left(\prod_{\breve{\sigma}_{n-1} \subset \partial \breve{\sigma}_n} CZ_{\breve{\sigma}_{n-1},\breve{\sigma}_n} \right).$$

$(d, n) = (3,1)$

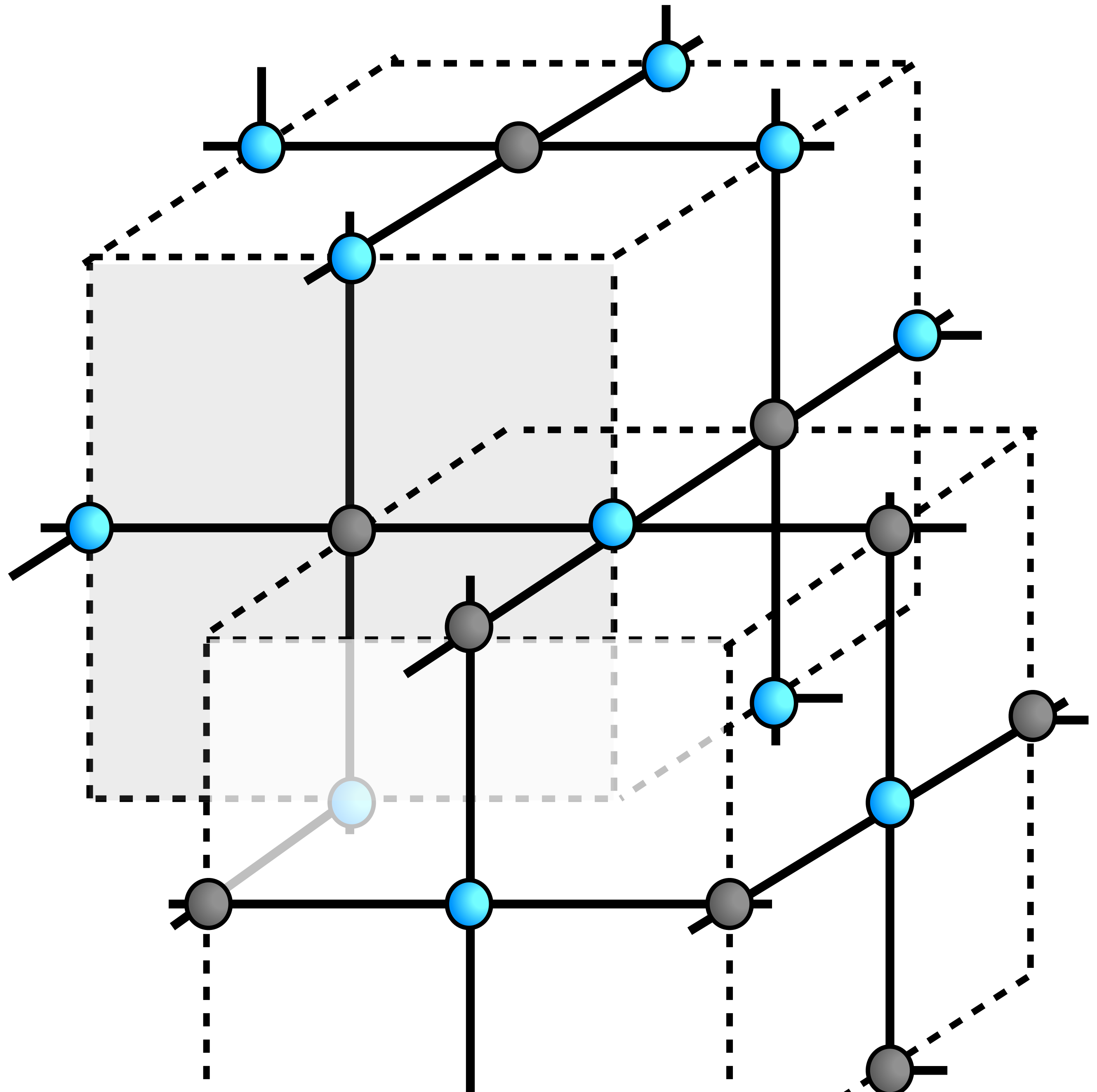


0-cell $\breve{\sigma}_0$

1-cell $\breve{\sigma}_1$

$(d, n) = (3,2)$

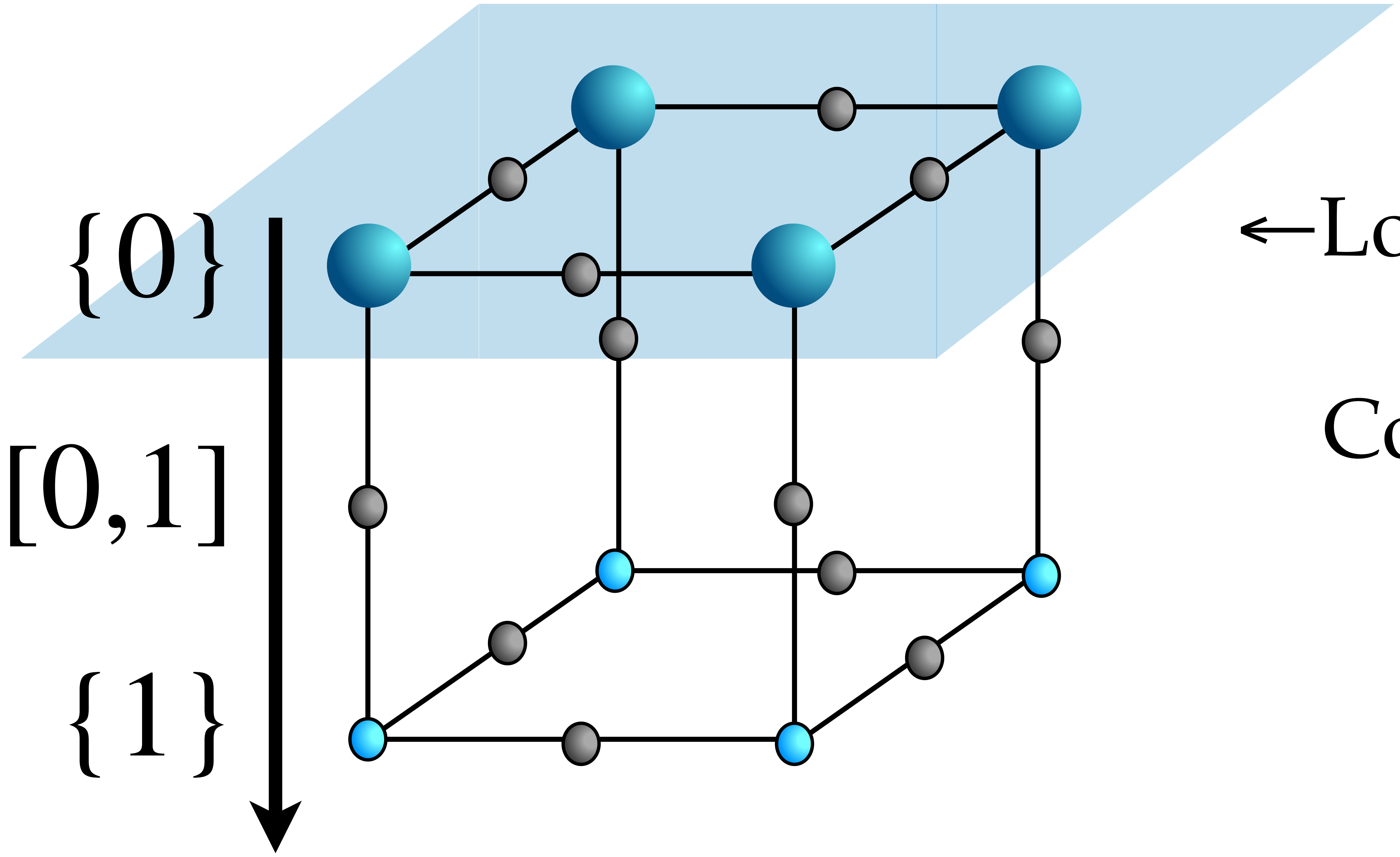
[Raussendorf Bravyi Harrington (2007)]



1-cell $\breve{\sigma}_1$

2-cell $\breve{\sigma}_2$

MBQS: simulating $M_{(3,1)}$ on $\mathrm{gCS}_{(3,1)}$

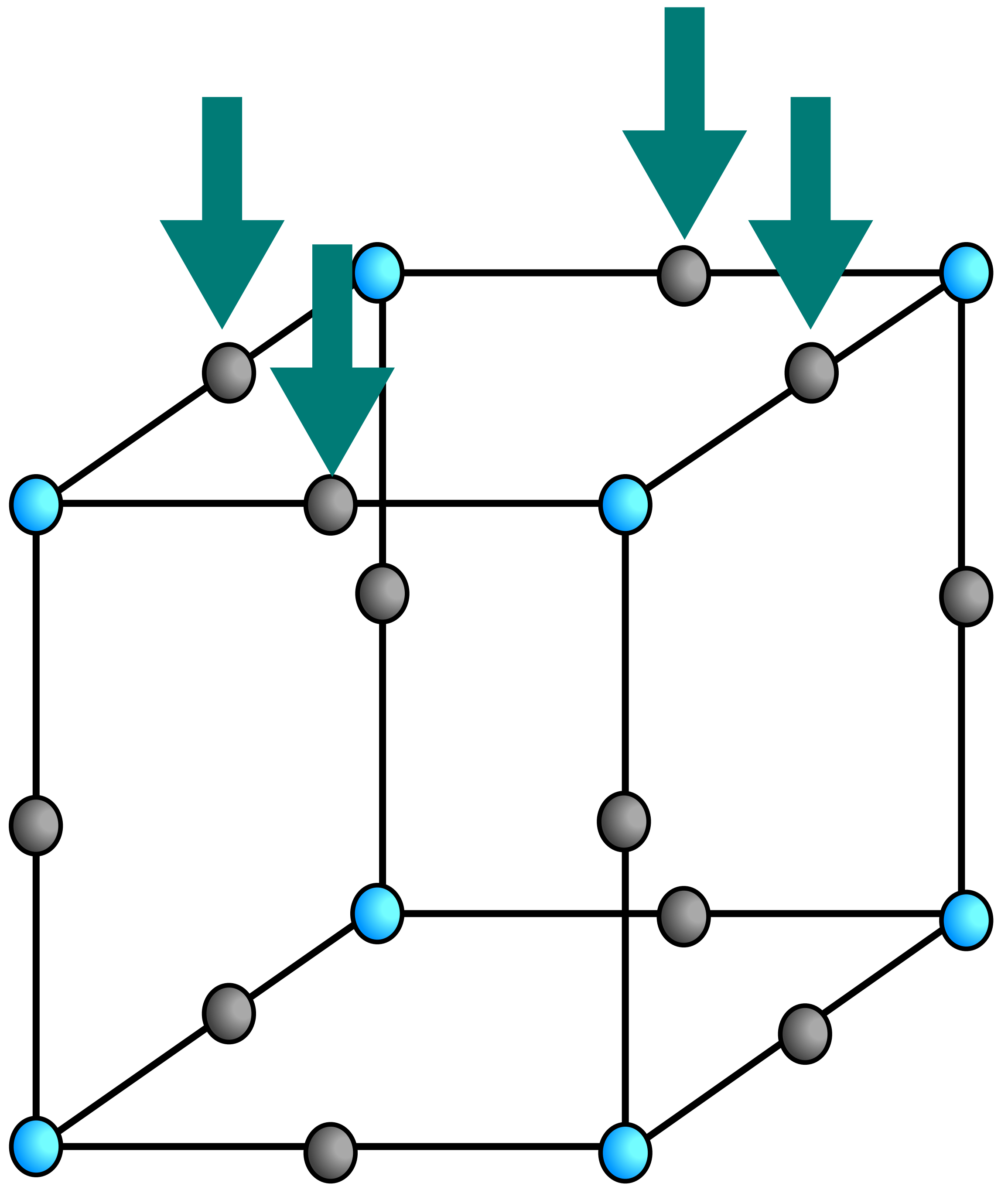


x_3-direction
="time" in the simulated world

←Load a 2d initial state $|\psi(0)\rangle_{\mathrm{bdry}}$ at $x_3 = 0$.

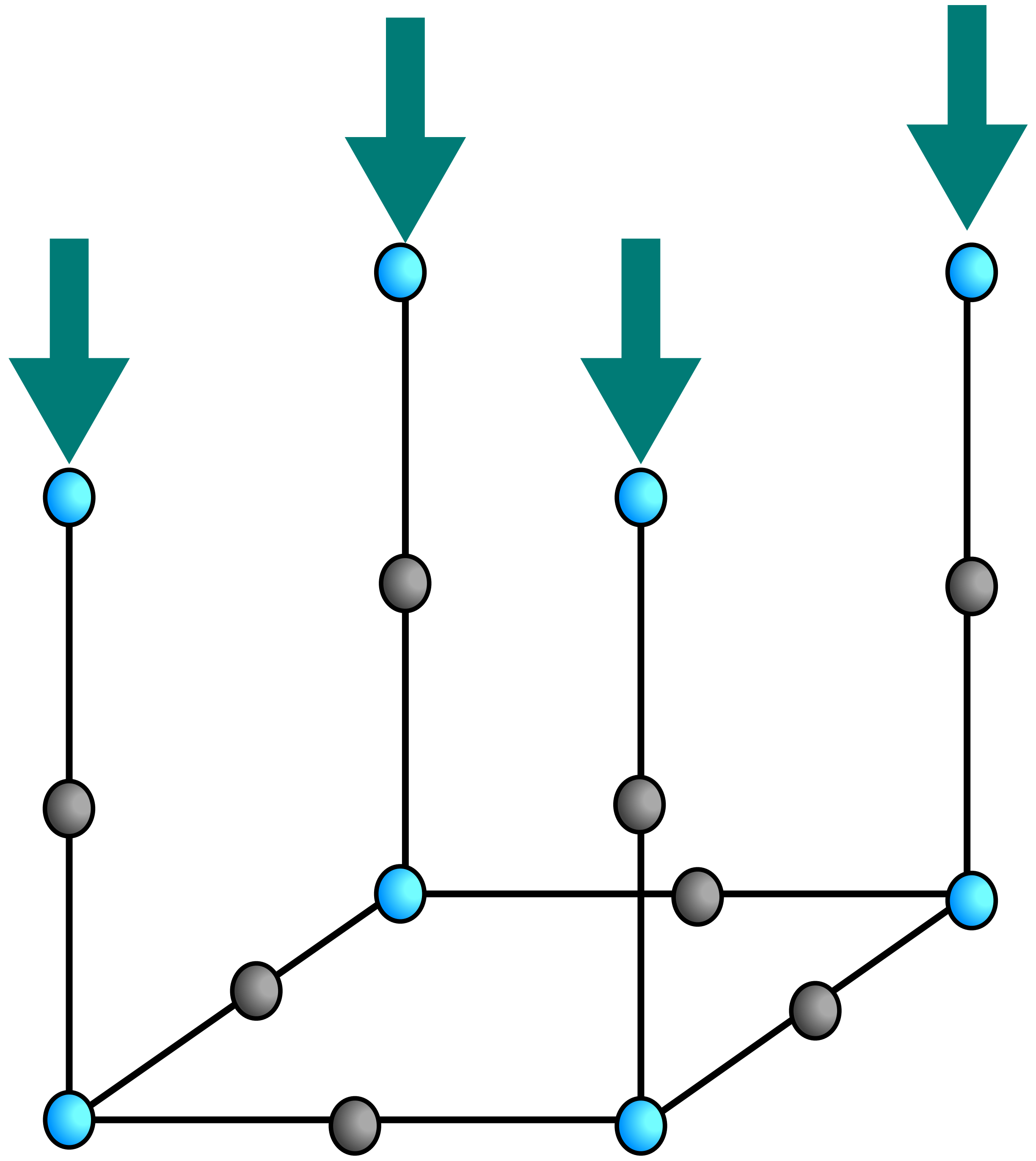
Couple it to the rest of the resource state.

MBQS: simulating $M_{(3,1)}$ on $\mathrm{gCS}_{(3,1)}$



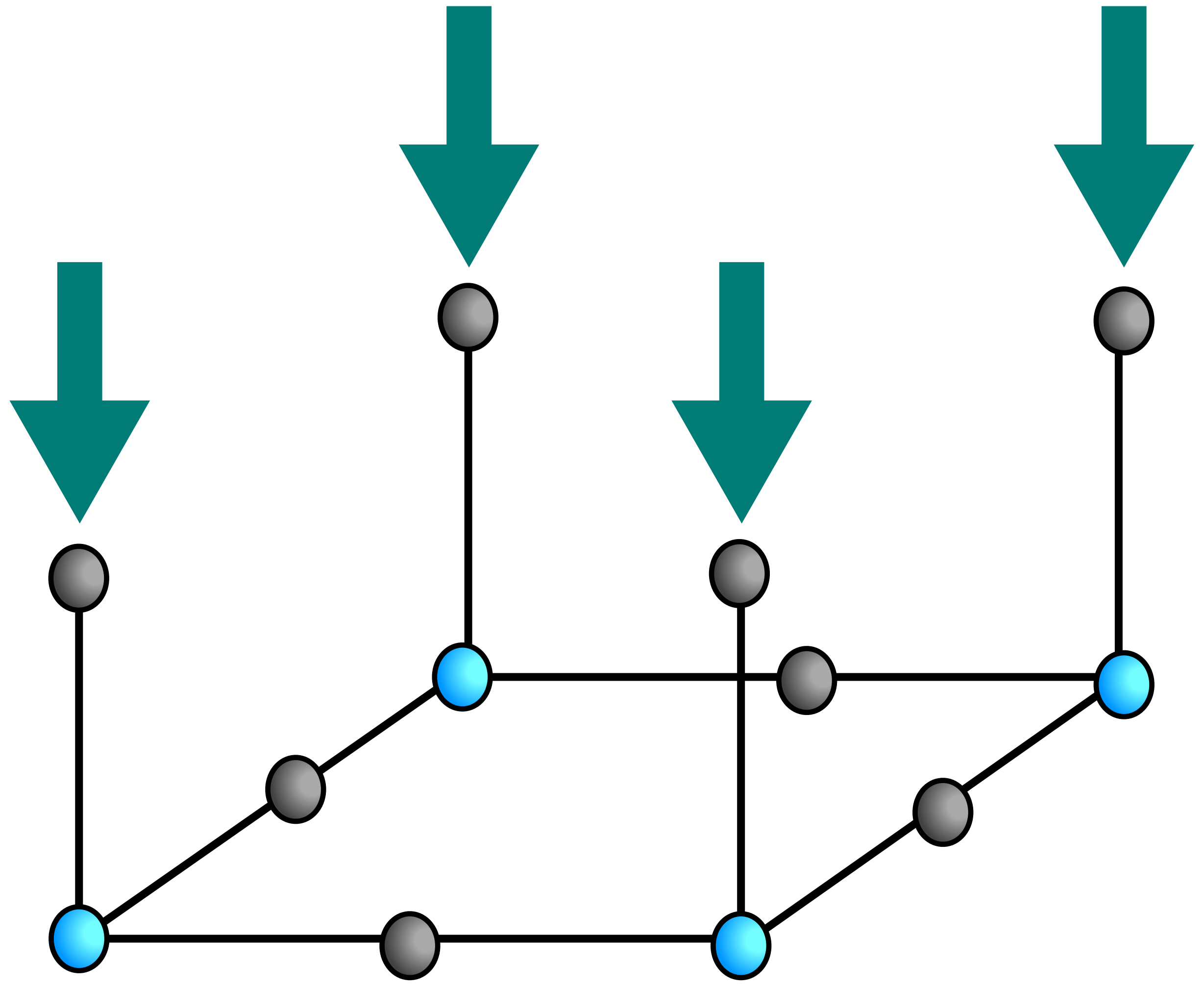
$\breve{\sigma}_1 = \sigma_1 \times \{j\}$

$$\prod_{\sigma_1} e^{-i\xi_1 Z(\partial\sigma_1)}$$



$\breve{\sigma}_0 = \sigma_0 \times \{j\}$

teleported to $[j, j+1]$

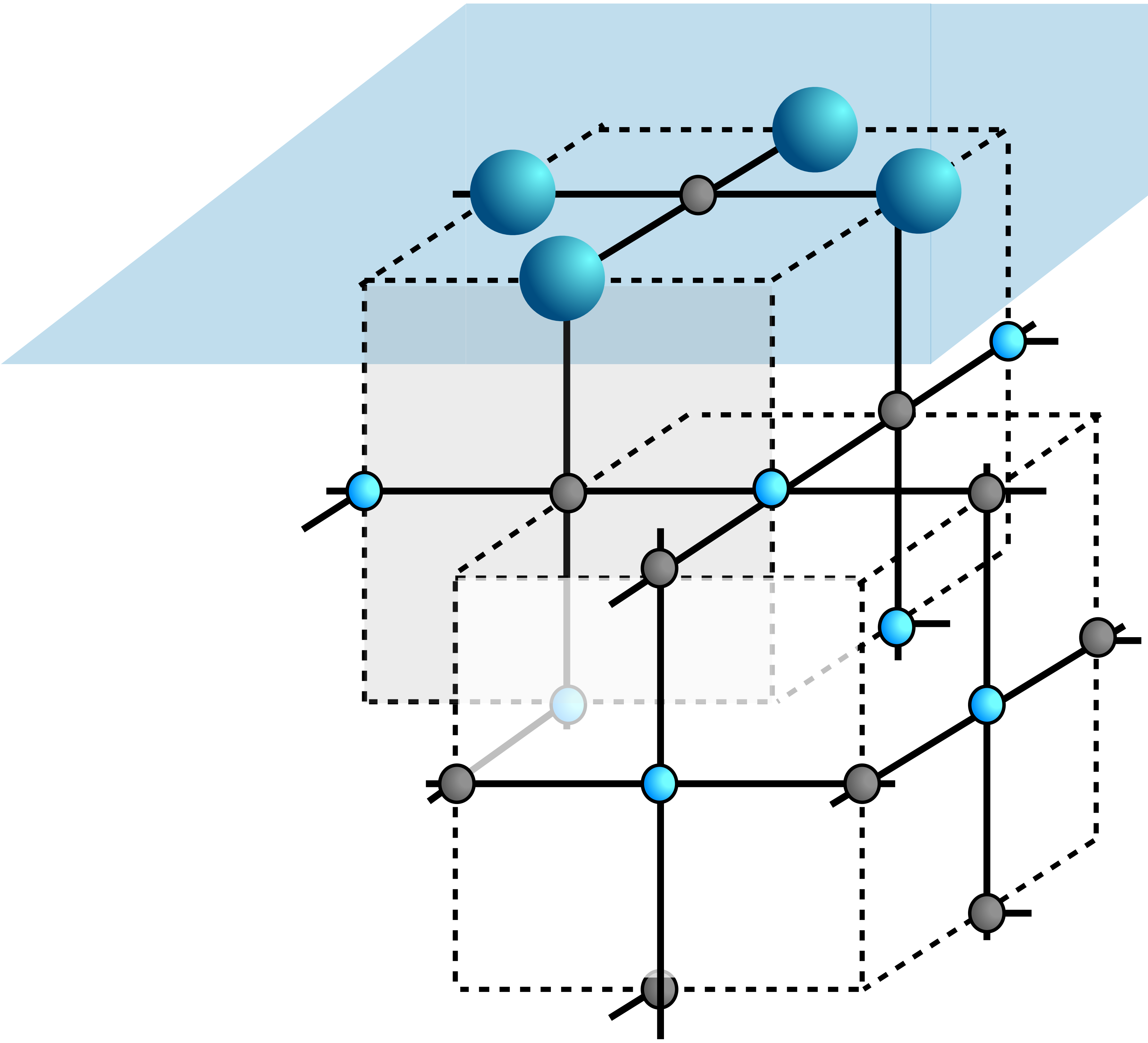


$\breve{\sigma}_1 = \sigma_0 \times [j, j+1]$

$$\prod_{\sigma_0} e^{-i\xi_3 X(\sigma_0)}$$

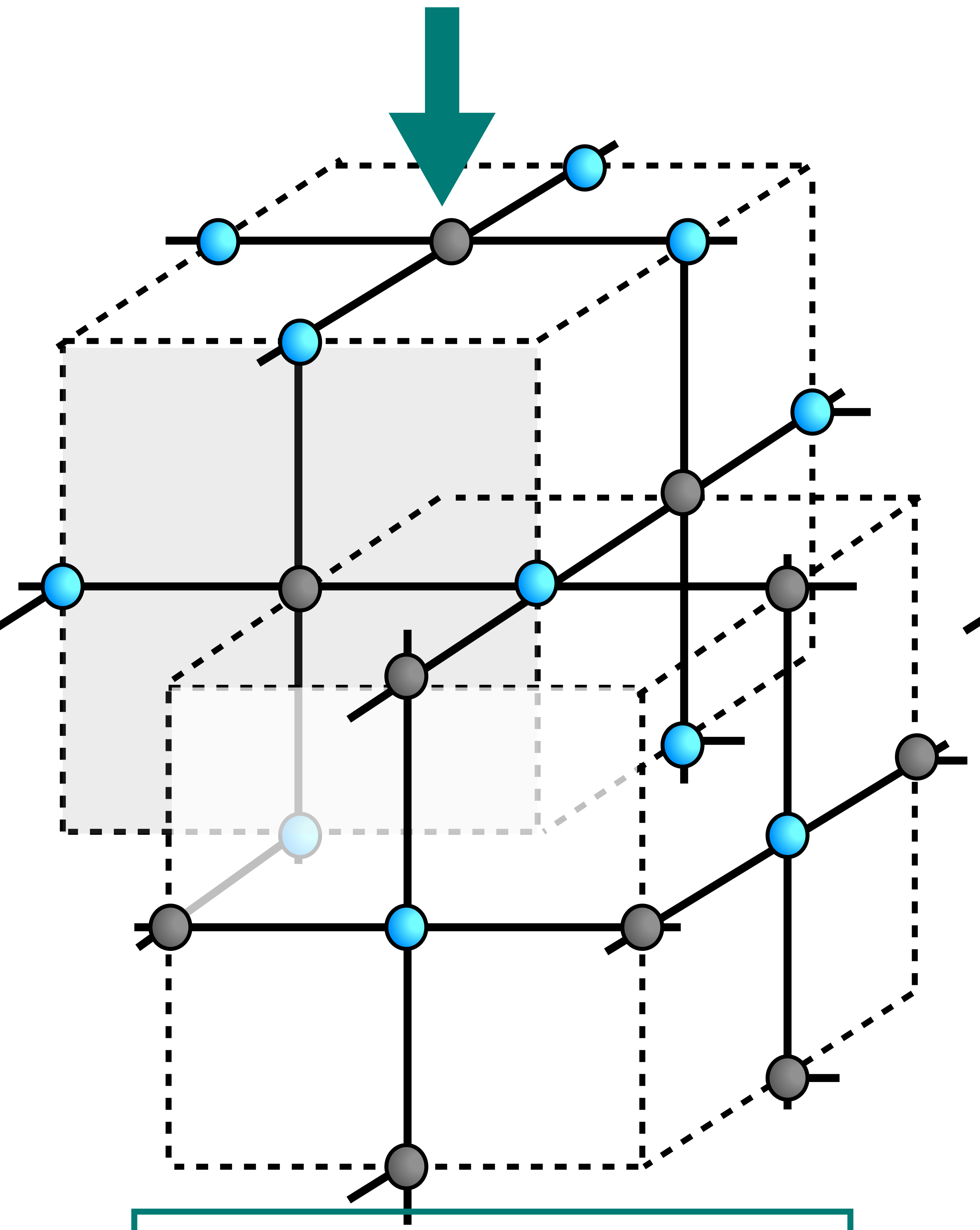
teleported to $\{j+1\}$

MBQS: simulating $M_{(3,2)}$ on gCS$_{(3,2)}$



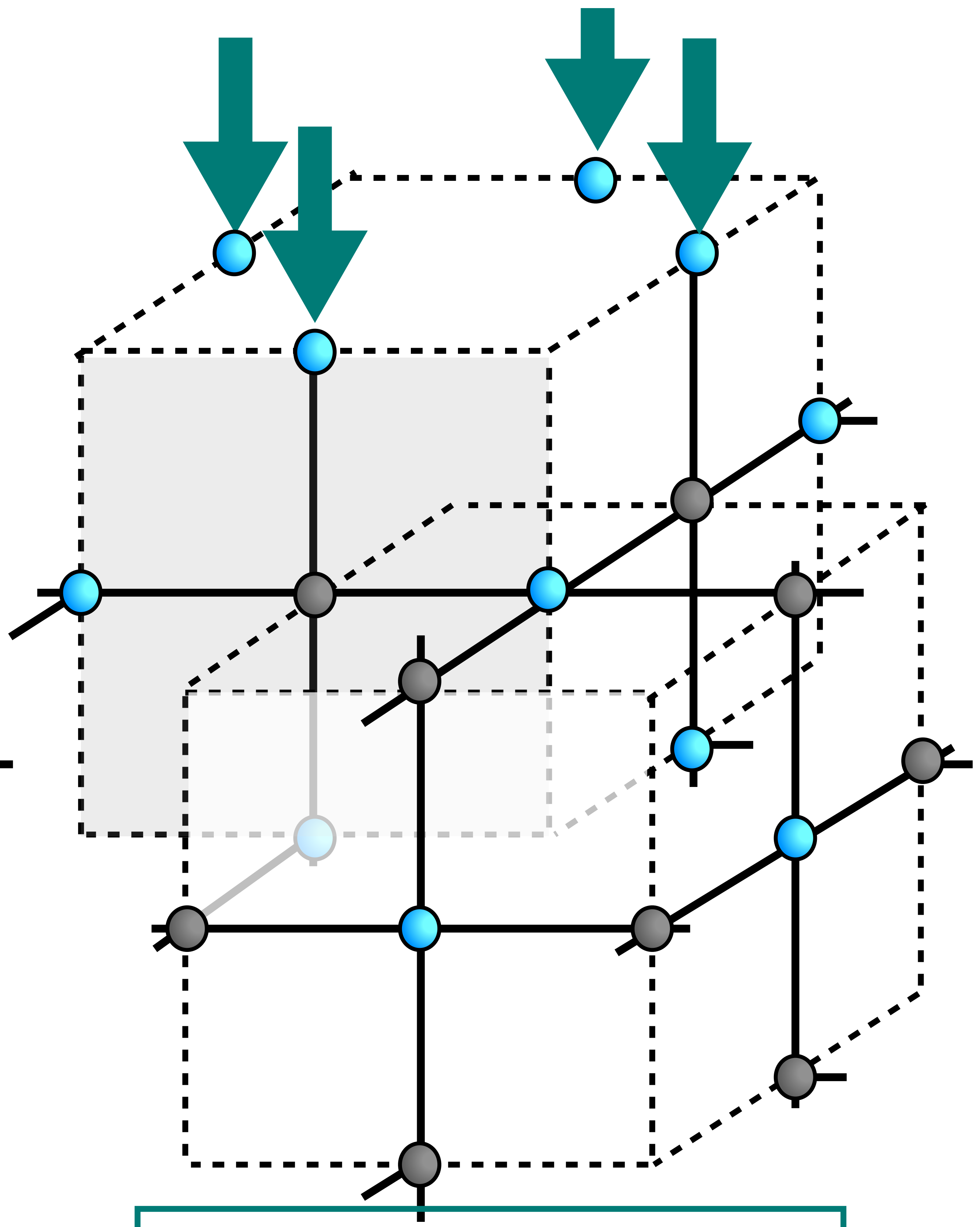
←Load a 2d initial state $|\psi(0)\rangle_{\text{bdry}}$ of the gauge theory

MBQS: simulating $M_{(3,2)}$ on $\mathrm{gCS}_{(3,2)}$



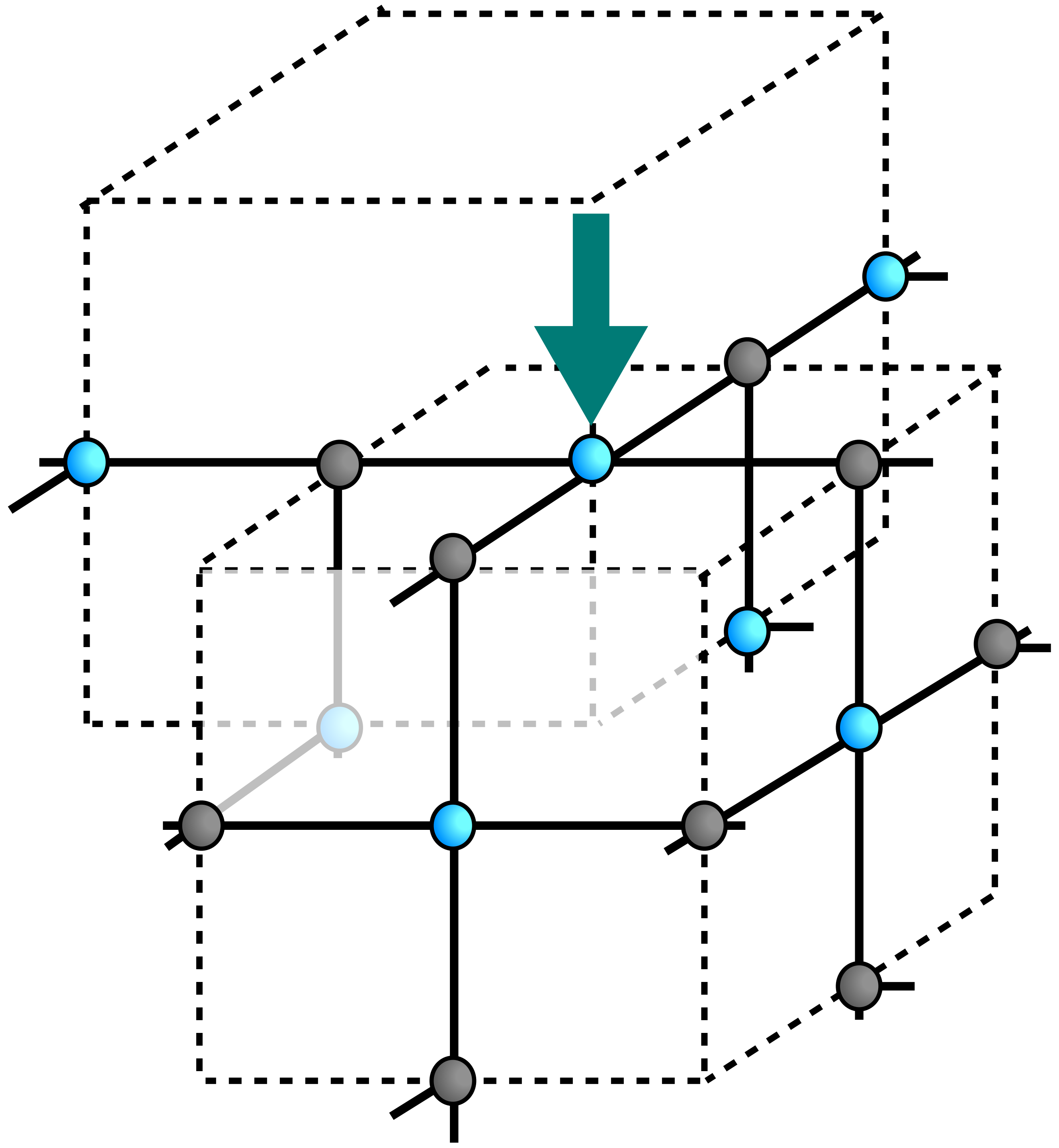
$\breve{\sigma}_2 = \sigma_2 \times \{j\}$

$$\prod_{\sigma_2} e^{-i\xi_1 Z(\partial\sigma_2)}$$



$\breve{\sigma}_1 = \sigma_1 \times \{j\}$

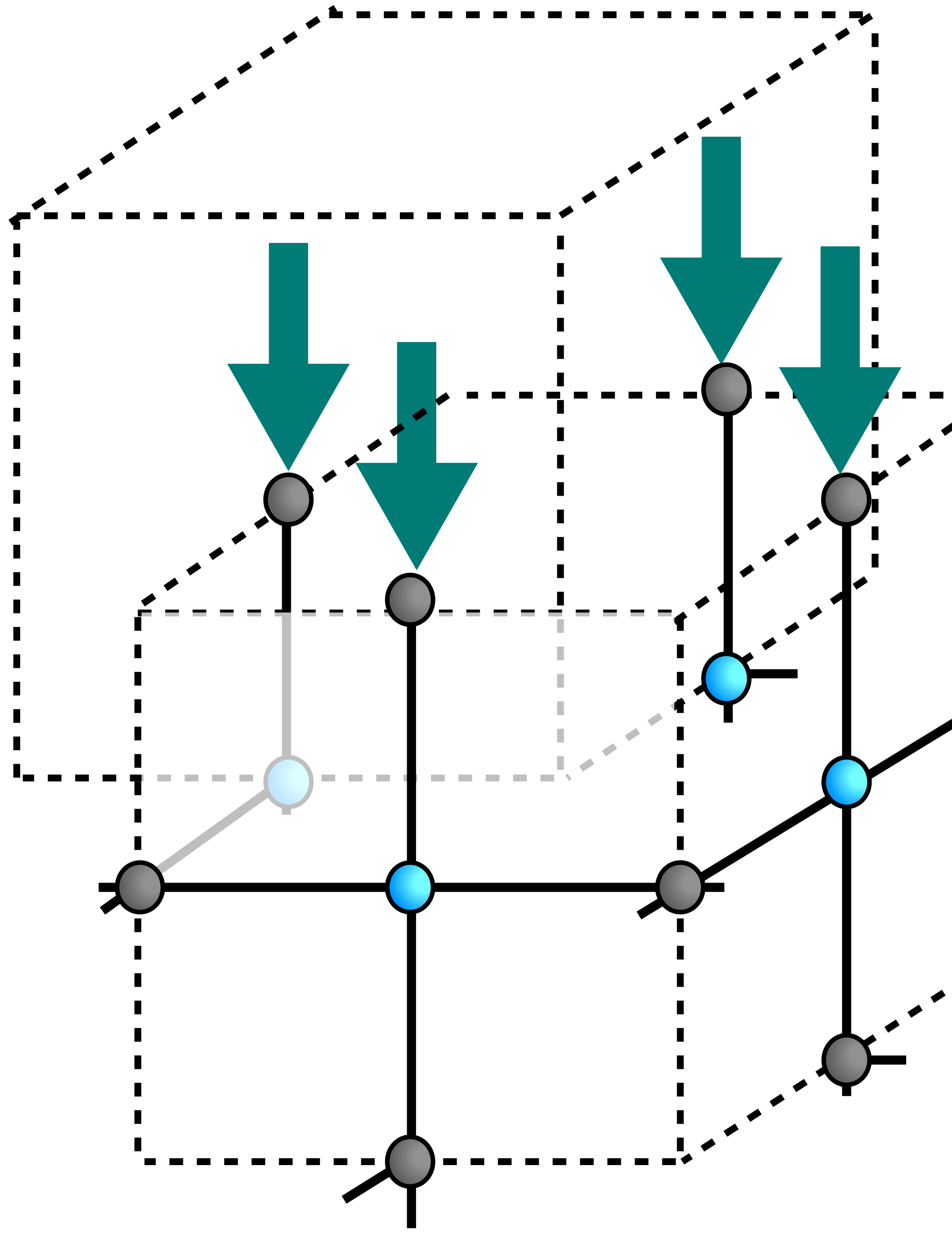
teleported to $[j, j+1]$



$\breve{\sigma}_1 = \sigma_0 \times [j, j+1]$

Gauss law check.

(Come back to this later)



$\breve{\sigma}_2 = \sigma_1 \times [j, j+1]$

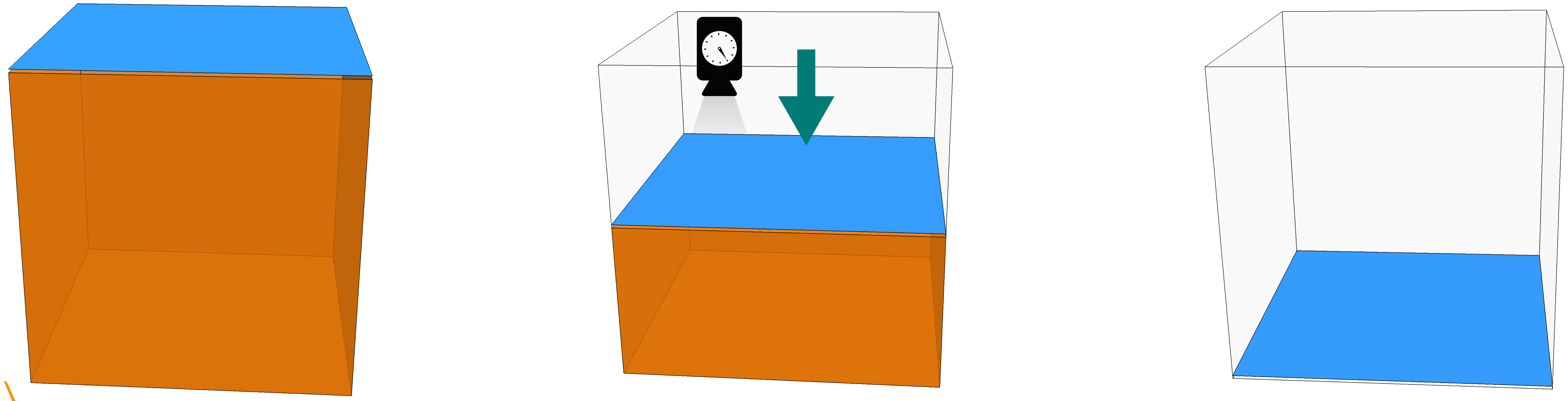
$$\prod_{\sigma_1} e^{-i\xi_4 X(\sigma_1)}$$

teleported to $\{j+1\}$

MBQS: simulating $M_{(d,n)}$ on $\mathrm{gCS}_{(d,n)}$

A state in $M_{(d,n)}$

Single-qubit measurements



$|\,\mathrm{gCS}_{(d,n)}\rangle$

MBQS: simulating $M_{(d,n)}$ on gCS$_{(d,n)}$

Ex. $M_{(3,2)}$ gauge theory

- We consider a faulty resource state $|\mathrm{gCS}^E\rangle = Z(\breve{e}_1)X(\breve{e}'_1)Z(\breve{e}_2)X(\breve{e}'_2)|\mathrm{gCS}\rangle$
- Perfect (non-faulty) measurement

The 2d simulated state at $x_3 = j \ (t = j\delta t)$ looks like:

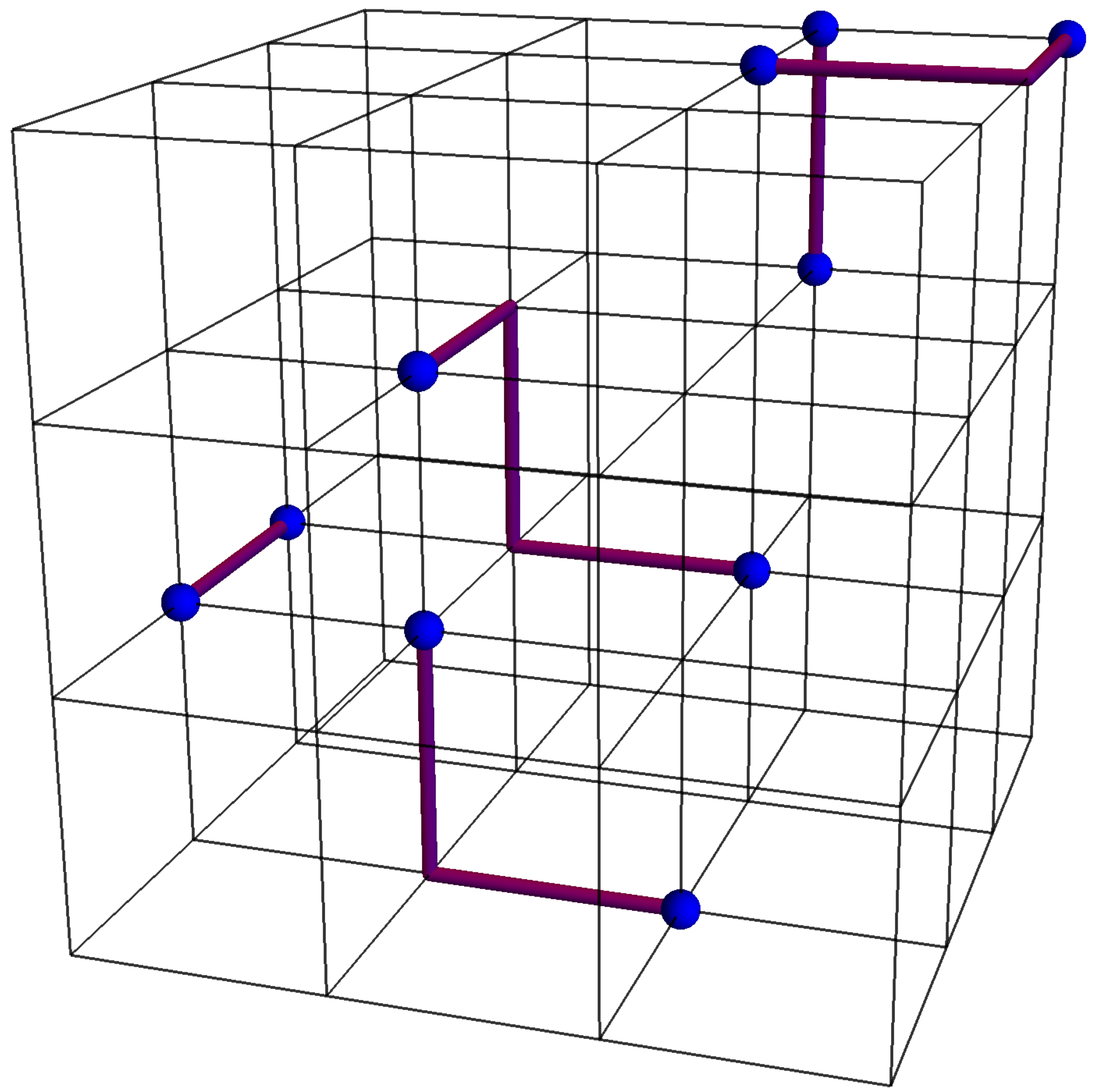
$$|\psi(t)\rangle = Z(e_1^{(j)})X(e_1^{\prime(j)})\left(\prod_k^j \Sigma^{(k)}\right) U^E(t)|\psi(0)\rangle$$

with $U^E(t)$ being Trotter evolution unitary with parameters $\tilde{\xi}_{1,4}$ being faulty.

$[Z(e_1^{(j)}), G(\sigma_0)] \neq 0$ The error chain $Z(e_1^{(j)})$ is caused by $Z(\breve{e}_1)$.

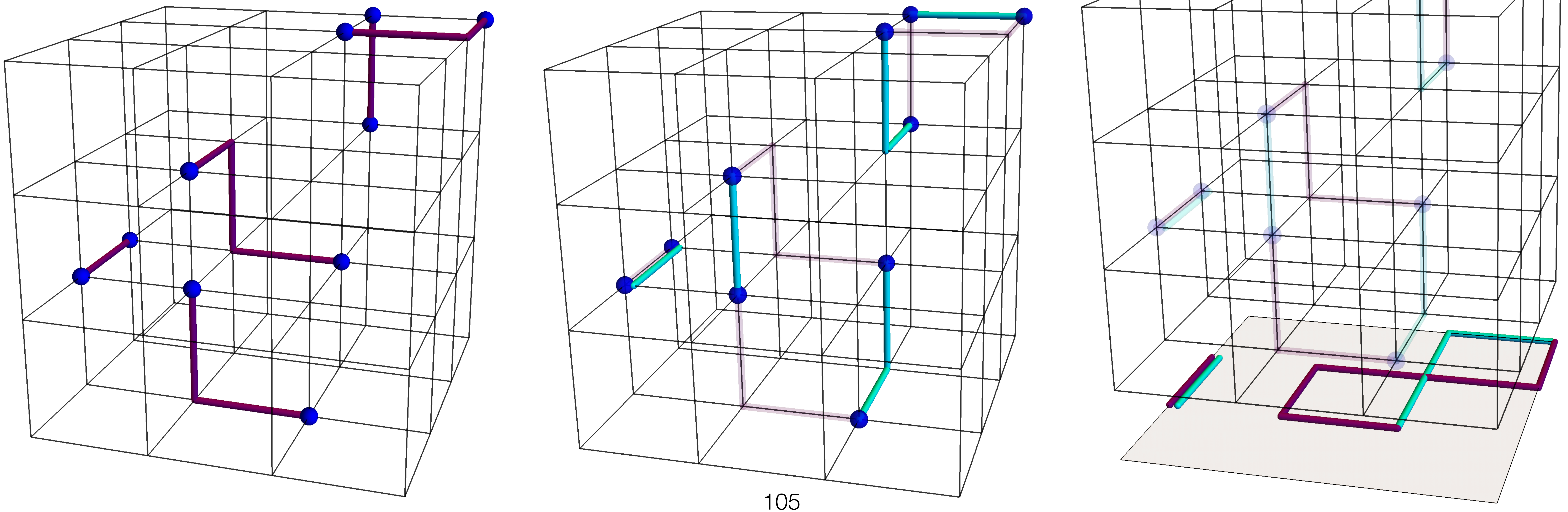
MBQS: simulating $M_{(d,n)}$ on gCS$_{(d,n)}$

- A symmetry of gCS: $|\mathrm{gCS}\rangle = X(\partial^* \breve{\sigma}_0)|\mathrm{gCS}\rangle$
- Error chain $Z(\breve{e}_1)$ flips the eigenvalue of $X(\partial^* \breve{\sigma}_0)$.
- In MBQS, the measurements at 1-chains are in X-basis.
- **→endpoints of $Z(\breve{e}_1)$ can be detected**

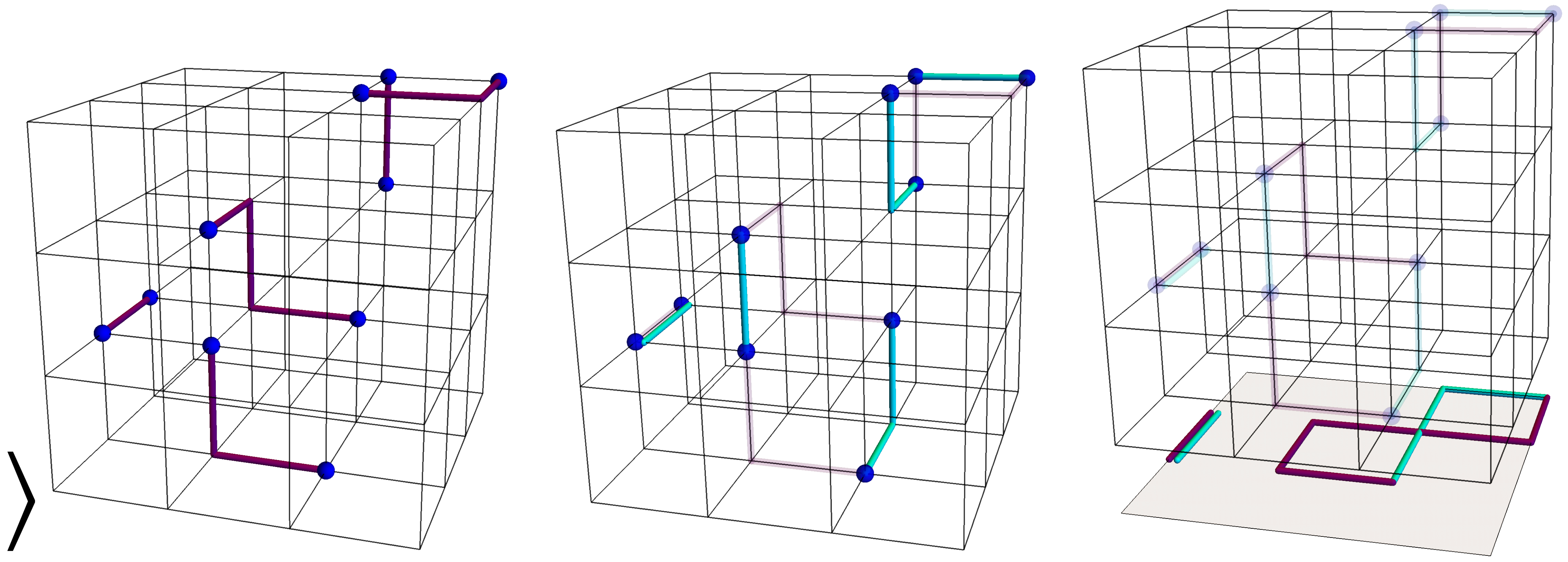


MBQS: simulating $M_{(d,n)}$ on $\mathrm{gCS}_{(d,n)}$

- A symmetry of gCS: $|\mathrm{gCS}\rangle = X(\partial^* \breve{\sigma}_0)|\mathrm{gCS}\rangle$
- Error chain $Z(\breve{e}_1)$ flips the eigenvalue of $X(\partial^* \breve{\sigma}_0)$.
- In MBQS, the measurements at 1-chains are in X-basis.
- **→endpoints of $Z(\breve{e}_1)$ can be detected**



MBQS: simulating $M_{(d,n)}$ on gCS$_{(d,n)}$



With correction, the 2d simulated state at $x_3 = j$ $(t = j\delta t)$ looks like:

$$|\psi(t)\rangle = Z(z_1^{(j)})X(e_1'^{(j)})\left(\prod_k^j \Sigma^{(k)}\right) U^{E+R}(t)|\psi(0)\rangle$$

with $z_1^{(j)}$ being $\partial z_1^{(j)} = 0$.

post-process $\Sigma^{(k)}$

$$|\psi(T)\rangle = Z(z_1^{(L_3)})X(e_1'^{(L_3)})U^{E+R}(T)|\psi(0)\rangle$$

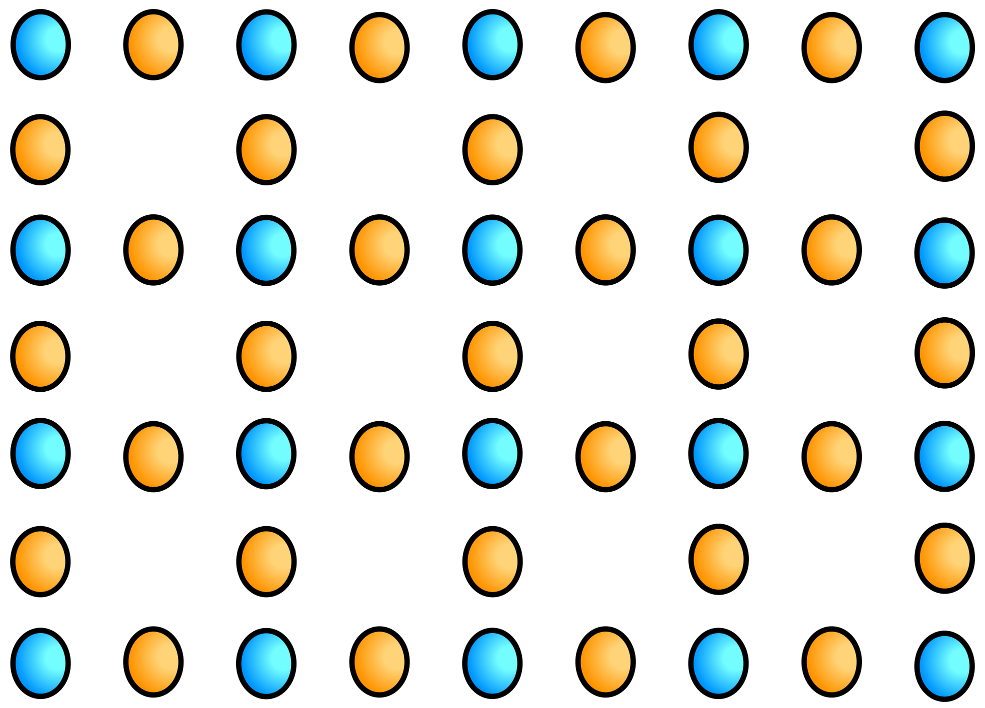
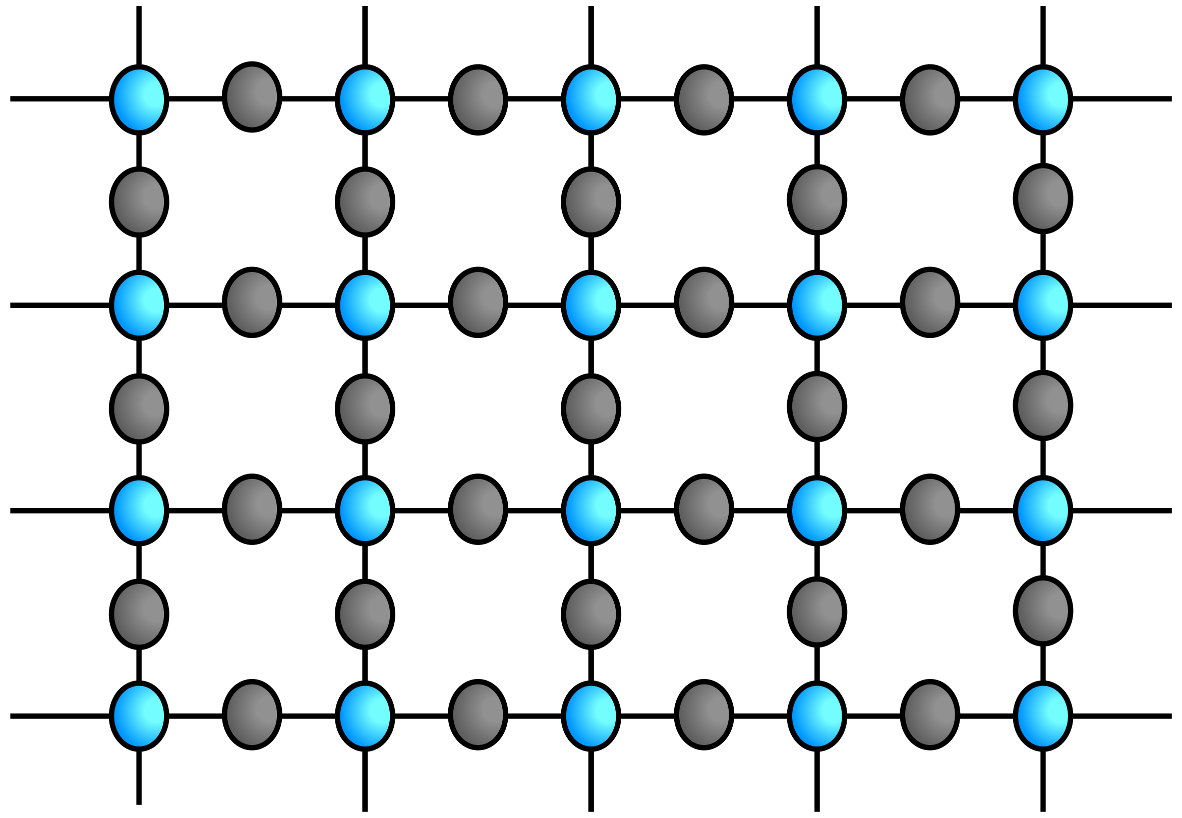
Gauss law is enforced:

$$G(\sigma_0)|\psi(T)\rangle = |\psi(T)\rangle$$

Overlap formula

Overlap formula

Our MBQS measurement pattern is related to the *overlap formula* below:

$$Z_{(2,1)} = \mathcal{N} \times \Big\langle$$  $$\Big|$$  $$\Big\rangle$$

2d ***classical*** Ising partition function

$\langle 0 | e^{-KX}$

$\langle + |$

$\mathrm{gCS}_{(2,1)}$

Resource state for (1+1)d transverse-field Ising model

It is a classical-quantum correspondence [Van den Nest-Dur-Briegel (2008)] relating a 2d quantum state and a 2d classical statistical model. See also [Lee-Ji-Bi-Fisher (2022)] [Matsuo-Fujii-Imoto (2014)].
The state $\langle 0 | e^{-KX}$ is different from $\langle 0 | e^{-i\xi X}$, which we used in MBQS, however.

Overlap formula

Let us check this formula.

$$\langle +|^V \bigotimes_{e\in E} \langle 0|e^{KX_e}|\mathrm{gCS}\rangle$$

$$\langle +|^V \bigotimes_{e\in E} \langle 0|e^{KX_e}\Big(\prod_{e\in E}\prod_{v\subset e} CZ_{e,v}\Big)|+\rangle^V|+\rangle^E$$

$$= \langle +|^V\langle 0|^E\Big(\prod_{e\in E}\prod_{v\subset e} CZ_{e,v}\Big)\prod_{e\in E} e^{KX_e\prod_{v\subset e}Z_v}|+\rangle^V|+\rangle^E$$

$$= \langle +|^V\langle 0|^E\prod_{e\in E} e^{(+1)K\prod_{v\subset e}Z_v}|+\rangle^V|+\rangle^E$$

$$= \frac{1}{2^{|E|/2}}\langle +|^V\prod_{e\in E} e^{(+1)K\prod_{v\subset e}Z_v}|+\rangle^V$$

Overlap formula

As Z is a diagonal operator in the computational basis, it reduces to evaluation of the exponential over all possible ± 1 configuration on vertices. We get

$$\frac{1}{2^{|E|/2}} \langle + |^V \prod_{e \in E} e^{(+1)K \prod_{v \subset e} Z_v} | + \rangle^V$$

$$= \frac{1}{2^{|E|/2} 2^{|V|}} \sum_{\{s_v = \pm 1\}_{v \in V}} \prod_{e \in E} e^{K \prod_{v \subset e} s_v}$$

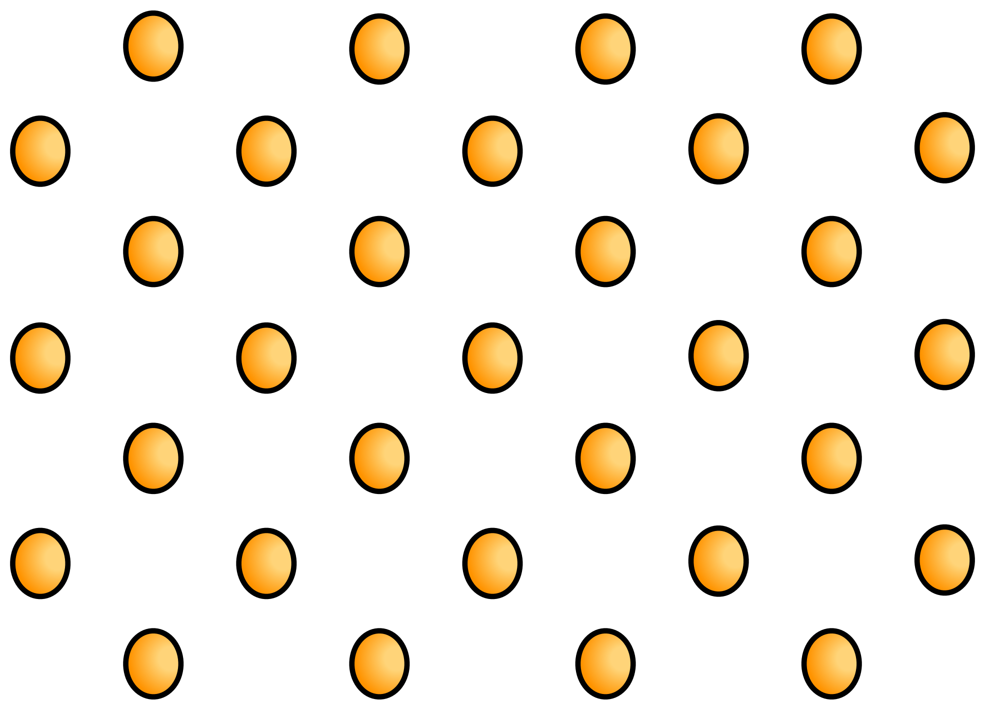
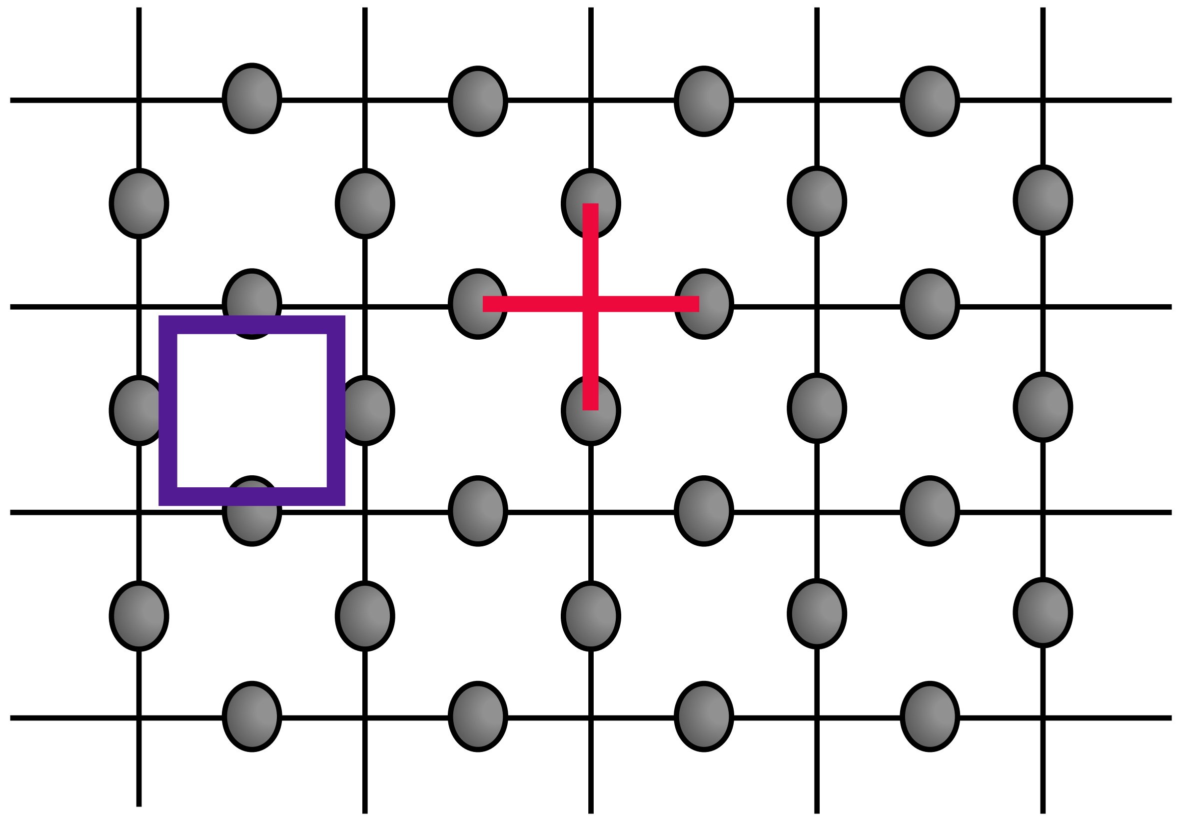
$$= \frac{1}{2^{|E|/2} 2^{|V|}} \sum_{\{s_v = \pm 1\}_{v \in V}} e^{K \sum_{e \in E} \prod_{v \subset e} s_v}$$

Thus we have

$$\langle + |^V \bigotimes_{e \in E} \langle 0 | e^{K X_e} | \mathrm{gCS} \rangle = \frac{1}{2^{|E|/2} 2^{|V|}} Z_{\mathrm{Ising}}(K)$$

Overlap formula

Rewriting it further,

$$Z_{(2,1)} = \mathcal{N} \times \Big\langle$$  $$\Big|$$  $$\Big\rangle$$

2d ***classical*** Ising
partition function

$\langle 0 | e^{-KX}$

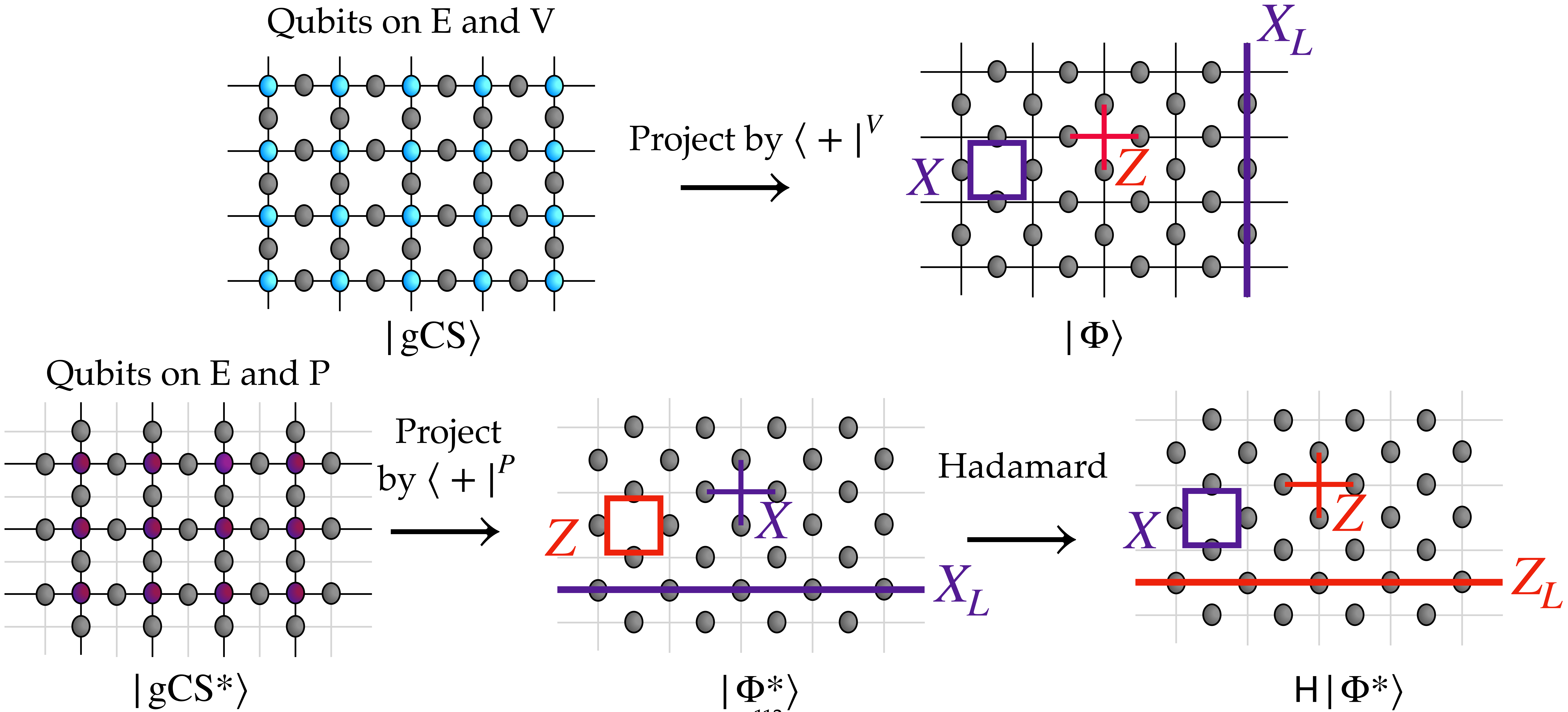
Toric code

= partially “measuring” out $\mathrm{gCS}_{(2,1)}$

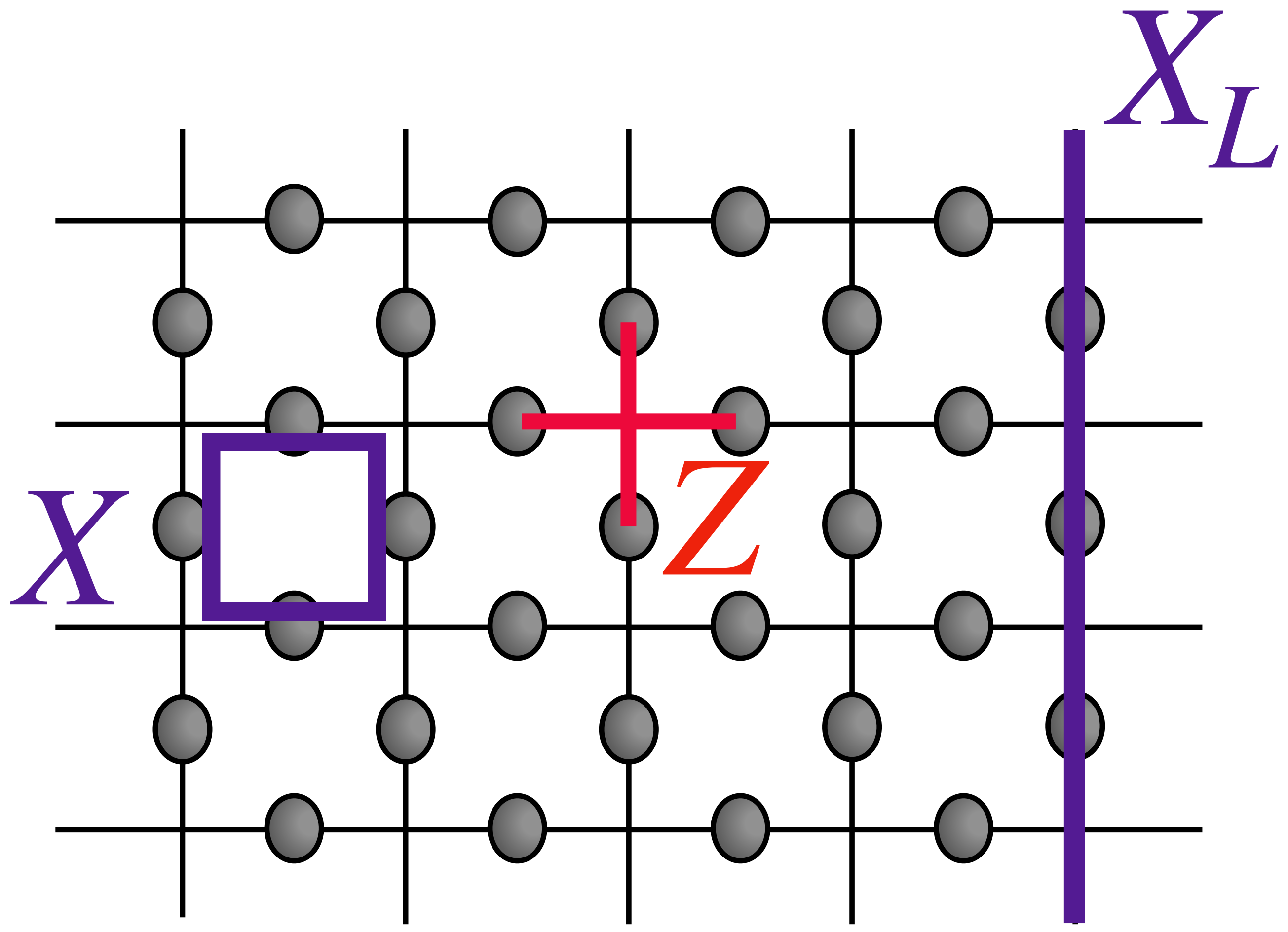
This is a ‘map’ from a topologically ordered state to a classical partition function.
In condensed matter physics, this type of relation is called a strange correlator.

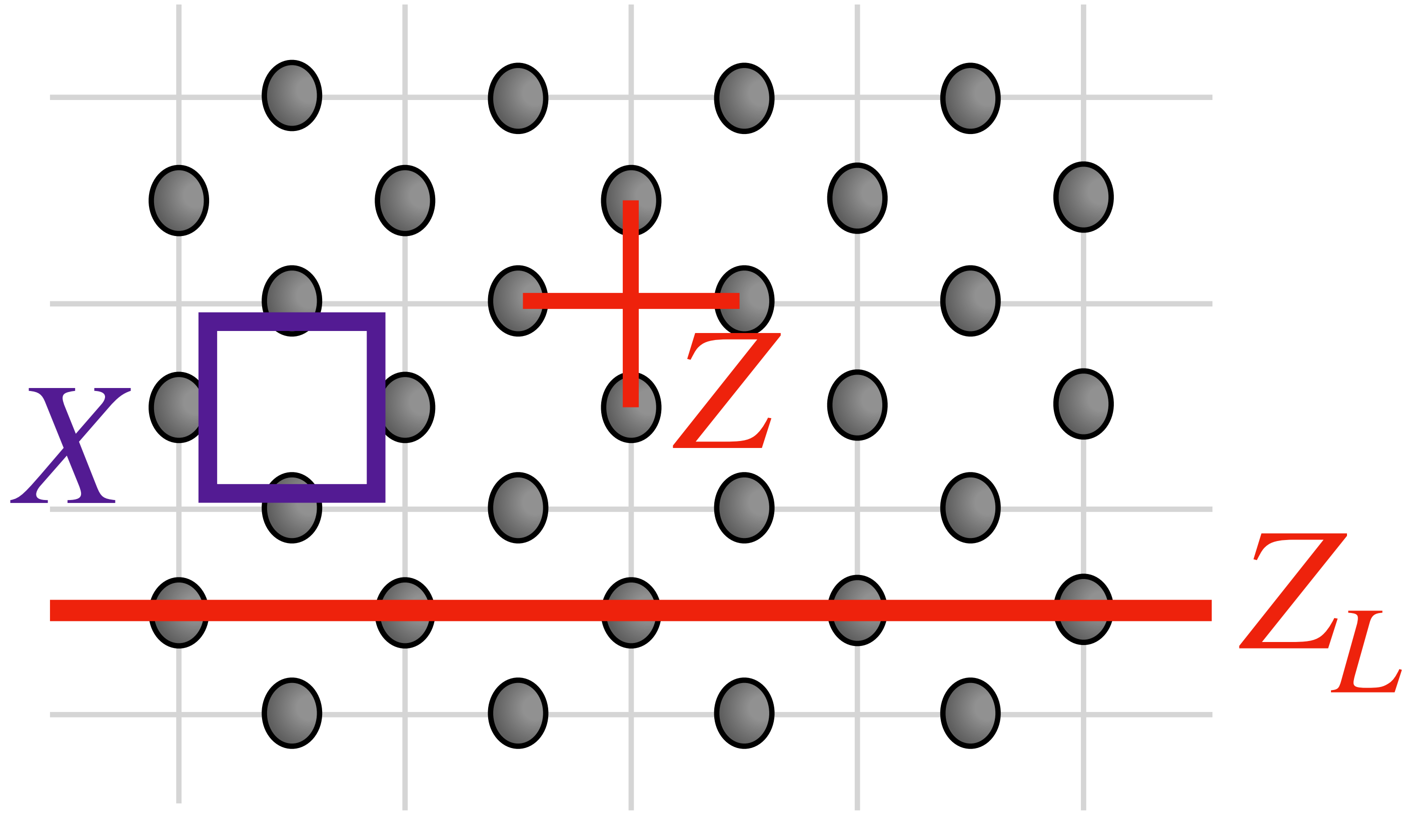
[Bal et al., Phys. Rev. Lett. 121, 177203 (2018)]

Overlap formula



Overlap formula





- The state $|\Phi\rangle$ is stabilized by $X_L|\Phi\rangle = |\Phi\rangle$
- The state $|\Phi^*\rangle$ is stabilized by $Z_L|\Phi^*\rangle = |\Phi^*\rangle$
- X_L and Z_L anti-commute on a torus.

The precise relation is:

$$\mathsf{H}|\Phi^*\rangle = \frac{1}{H_1(T^2,\mathbb{Z}_2)} \sum_{[\ell]\in H_1(T^2,\mathbb{Z}_2)} Z_\ell|\Phi\rangle$$

Note:

$$X_L|\mp\rangle = |\mp\rangle, \quad Z_L|\bar{0}\rangle = |\bar{0}\rangle, \quad |\mp\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\rangle + |\bar{1}\rangle\right)$$

Overlap formula

We obtained:

$$\mathsf{H}|\Phi^*\rangle = \frac{1}{H_1(T^2,\mathbb{Z}_2)} \sum_{[\ell]\in H_1(T^2,\mathbb{Z}_2)} Z_\ell |\Phi\rangle \,.$$

There's an identity $\langle 0| e^{KX}\mathsf{H} = \sqrt{\sinh(K)}\langle 0| e^{K^*X}$ with $K^* = -\dfrac{1}{2}\log\tanh(K)$.

The identity

$$\langle 0| e^{KX} |\Phi^*\rangle = \langle 0| e^{KX}\mathsf{H}\cdot\mathsf{H}|\Phi^*\rangle$$

implies that

$$Z_{\mathrm{dual}}(K) \sim (\sinh K)^{|E|/2} \sum_{[\ell]\in H_1(T^2,\mathbb{Z}_2)} Z(K^*;\ell)$$

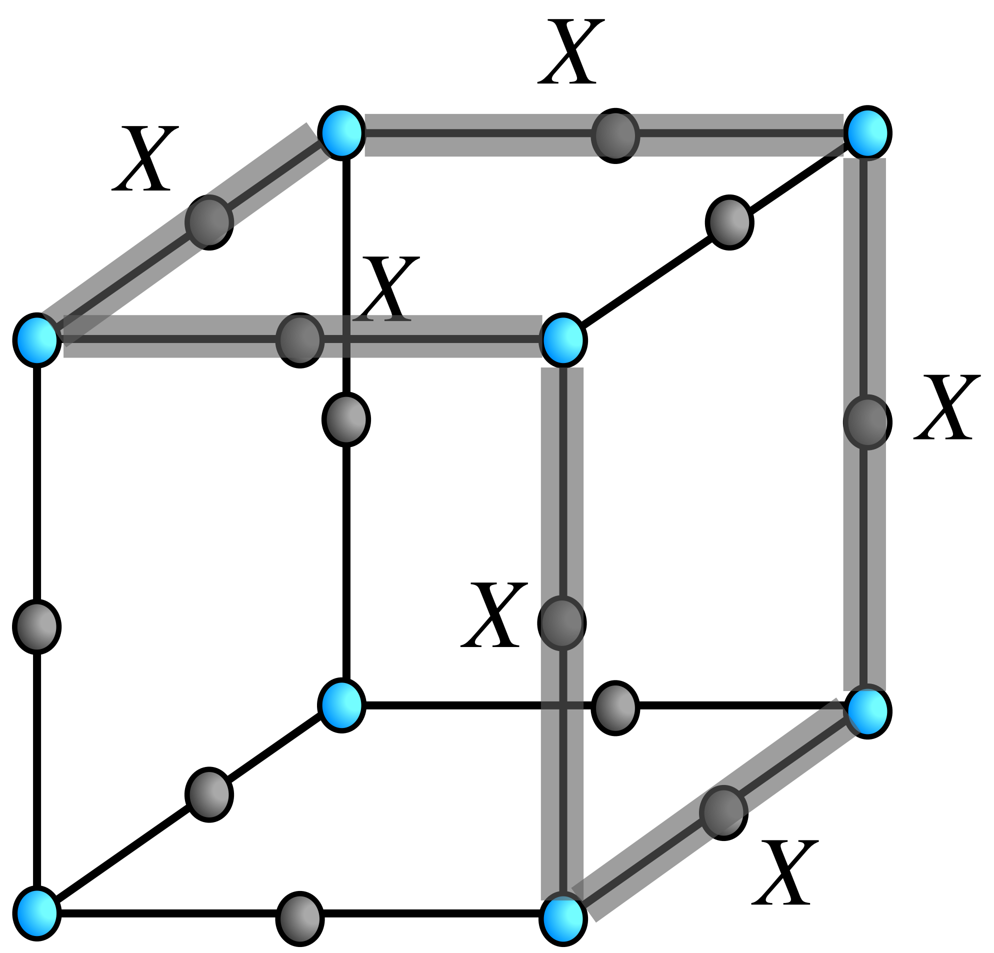
where $Z(K^*;\ell)$ is a twisted partition function of 2d classical partition function and $Z_{\mathrm{dual}}(K)$ is the Ising partition function on the dual square lattice.
The sign of the coupling constant is flipped along the line ℓ.

Aspects of symmetries I:
SPT

Higher-form symmetries in gCS

$(d, n) = (3,1)$

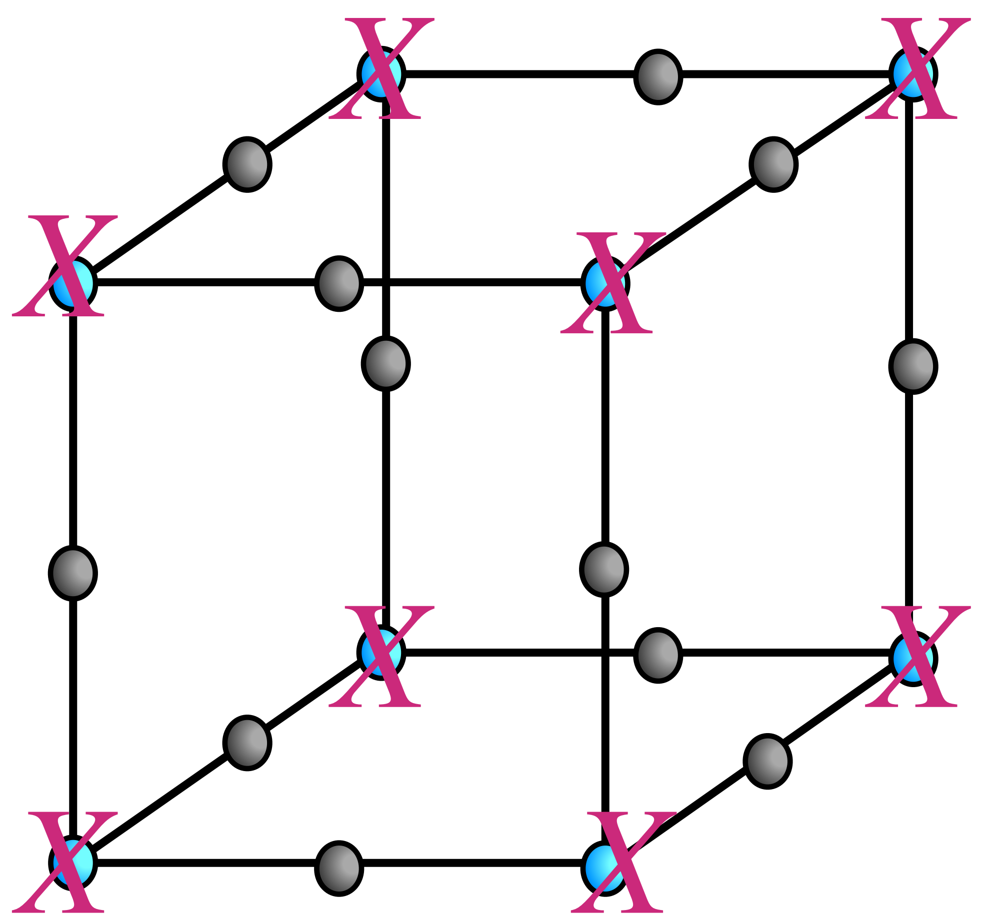
$(d - n) = 2$-form symmetry



$X(\breve{z}_1)$

$\partial \breve{z}_1 = 0$

$(n - 1) = 0$-form symmetry



$X(\breve{z}_3^*)$

$\partial^* \breve{z}_3^* = 0$

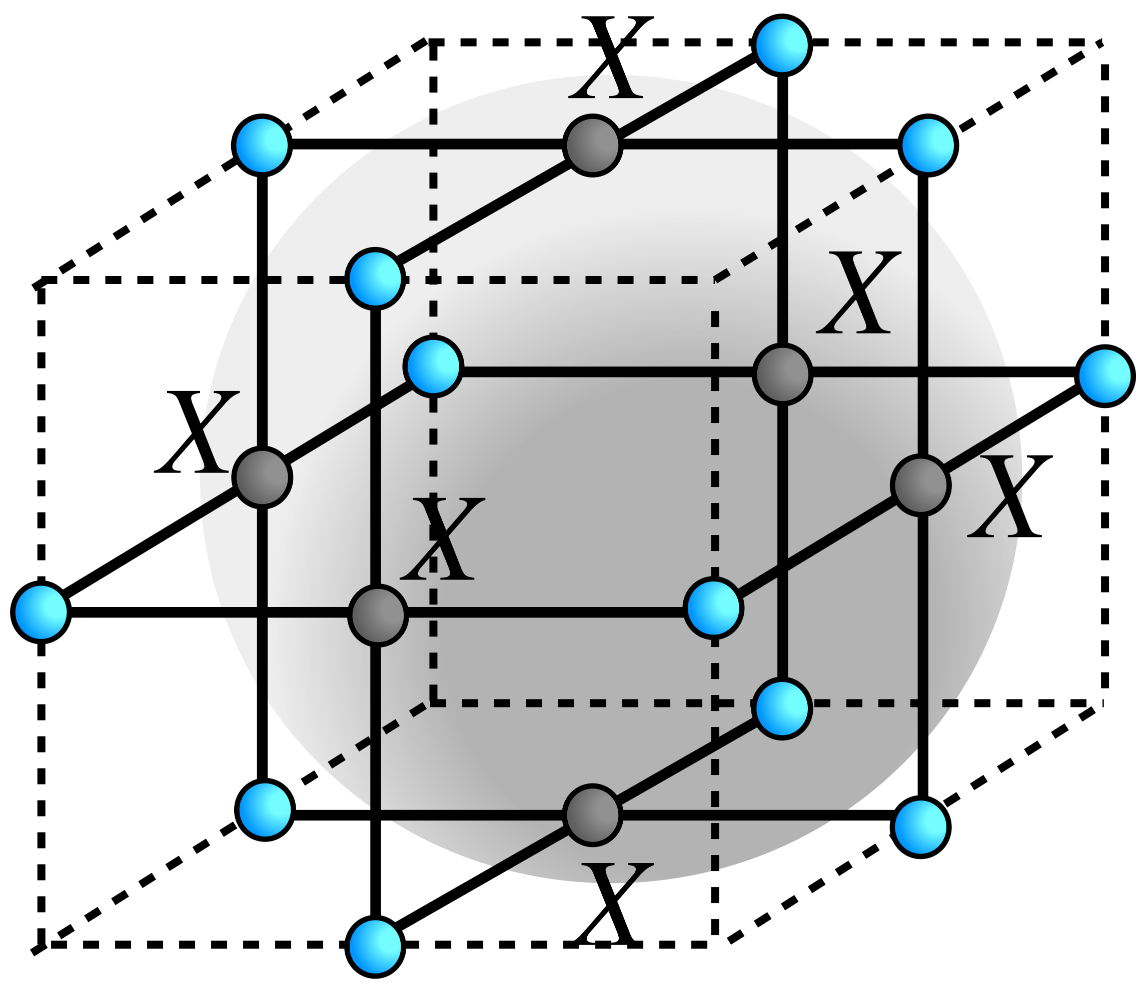
Higher-form symmetries in gCS

$(d, n) = (3,2)$

$(d - n) = 1$-form symmetry

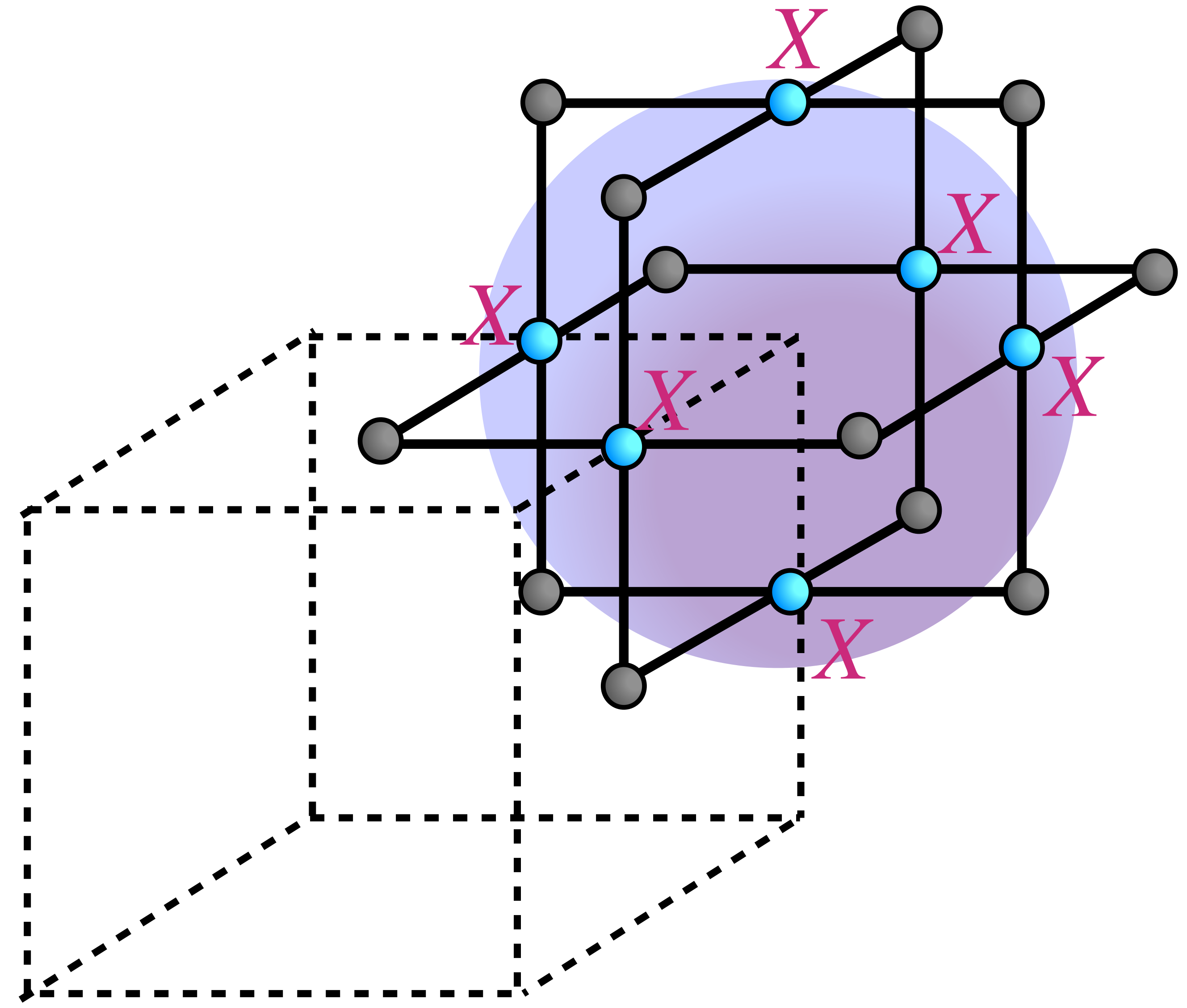
$(n - 1) = 1$-form symmetry

$X(\breve{z}_1)$



$\partial \breve{z}_1 = 0$

$X(\breve{z}_2^*)$



$\partial^* \breve{z}_2^* = 0$

Higher-form symmetries in gCS

$(d-n)$-form and $(n-1)$-form symmetry:

$$|\,\mathrm{gCS}\rangle = X(\breve{z}_n)\,|\,\mathrm{gCS}\rangle = X(\breve{z}^*_{d-n+1})\,|\,\mathrm{gCS}\rangle$$

with $M_{d-n} = \{\breve{z}_n \,|\, \partial\breve{z}_n = 0\}$, $M'_{n-1} = \{\breve{z}^*_{d-n+1} \,|\, \partial^*\breve{z}^*_{d-n+1} = 0\}$.

SPT order in gCS

gCS$_{(d,n)}$ has an SPT order protected by $(d-n)$-form and $(n-1)$-form $\mathbb{Z}_2$

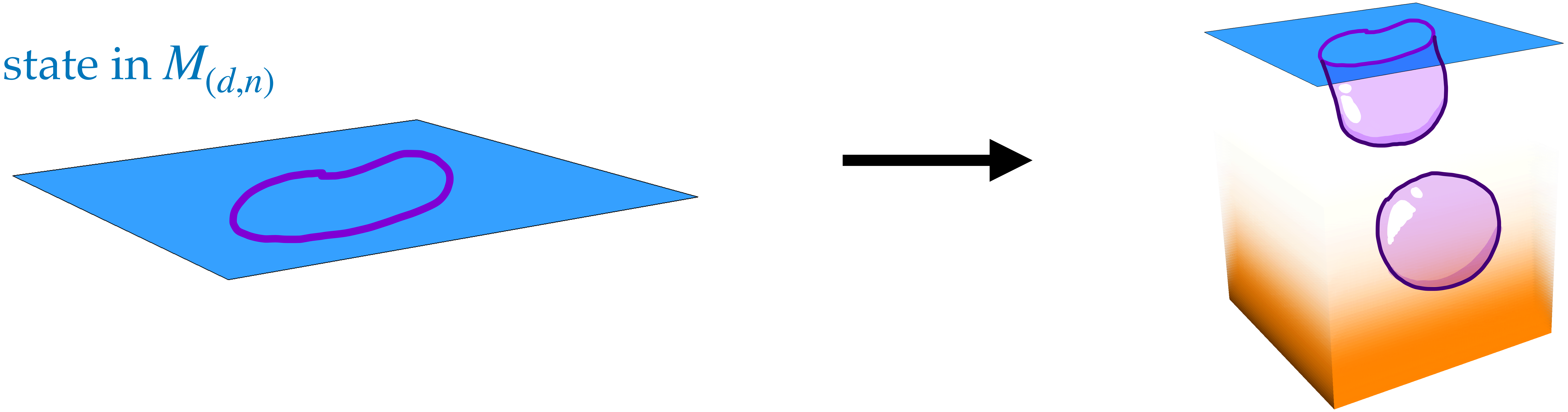
- Two symmetry generators act projectively at the boundaries of the lattice $\rightarrow$ SPT. Cf. [Yoshida (2016)] [Roberts-Kubica-Yoshida-Bartlett (2017)].
- The simulated state as an edge state of an SPT.

Appendix

Aspects of symmetries II: Holographic correspondence?

Bulk/boundary symmetries in MBQS

A state in $M_{(d,n)}$



Boundary symmetry generator $X(z^*_{d-n})$

Bulk symmetry generator $X(\breve{z}^*_{d-n+1})$ with $\partial^* \breve{z}^*_{d-n+1} = 0$ or $= z^*_{d-n}$.

(3,1) Ising — 0-form symmetry $X(z^*_2) = \prod_{v \in V} X_v$ → 0-form symmetry $X(\breve{z}^*_3) = \prod_{\breve{v} \in \breve{V}} X_{\breve{v}}$

(3,2) gauge — Electric 1-form symmetry $X(z^*_1)$ → 1-form symmetry $X(\breve{z}^*_2)$

Bulk/boundary symmetries in MBQS

Consider a d-dimensional Hamiltonian

$$H = -\sum Z(\partial\breve{\sigma}_n)\,,$$

which is symmetric under the transformation with the **global** $(n-1)$-form, $X(\breve{z}^*_{d-n+1})$.

Cluster state gCS:

It is described by the local stabilizer conditions:

$$X(\breve{\sigma}_n)Z(\partial\breve{\sigma}_n)\,|\,\mathrm{gCS}_{(d,n)}\rangle = X(\breve{\sigma}_{n-1})Z(\partial^*\breve{\sigma}_{n-1})\,|\,\mathrm{gCS}_{(d,n)}\rangle = |\,\mathrm{gCS}_{(d,n)}\rangle\,.$$

It can be seen as the ground state of the **gauged version** of the above Hamiltonian,

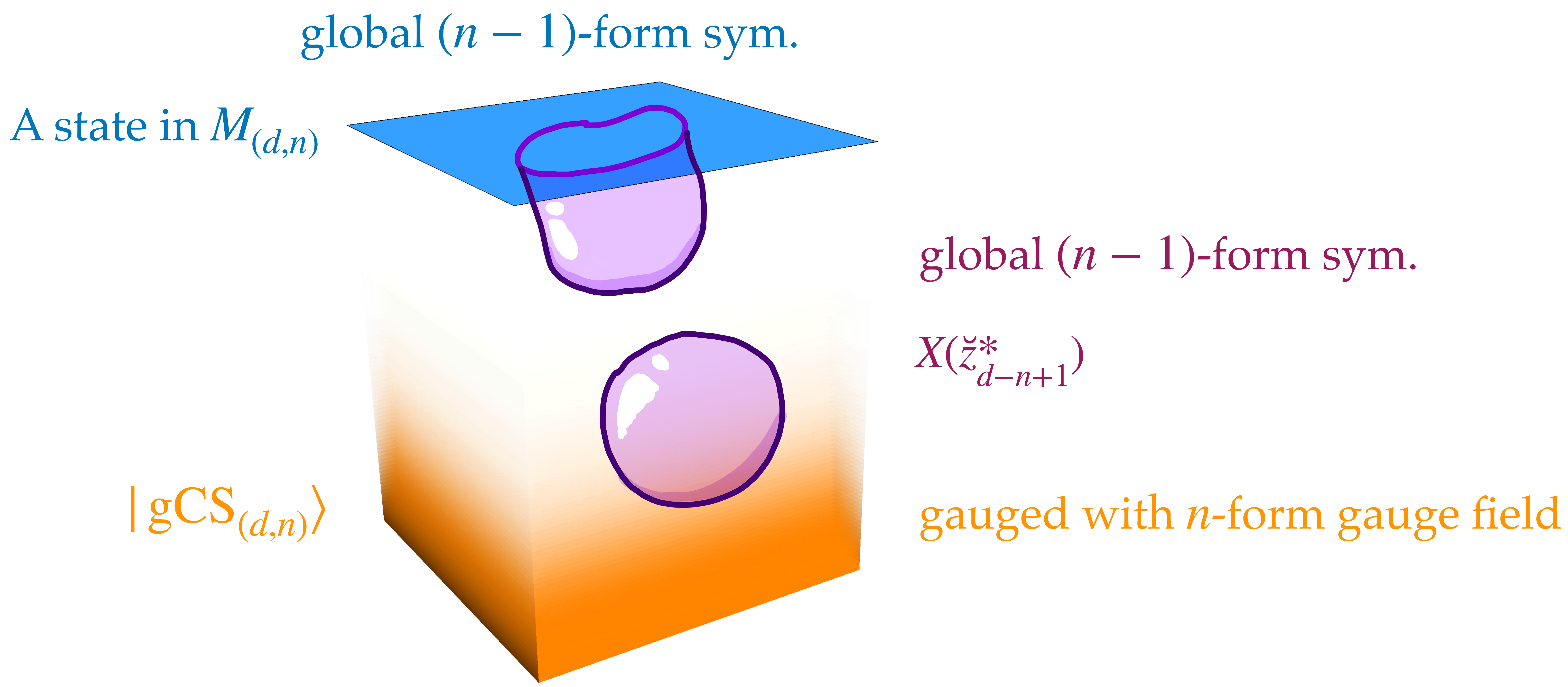
$$H_{\text{gauged}} = -\sum X(\breve{\sigma}_n)Z(\partial\breve{\sigma}_n)\,,$$

with the local gauge constraint $X(\breve{\sigma}_{n-1})Z(\partial^*\breve{\sigma}_{n-1}) = 1$ $(\forall\breve{\sigma}_{n-1})$.

(The global symmetry $X(\breve{z}^*_{d-n+1})$ is a product of local stabilizers $X(\breve{\sigma}_{n-1})Z(\partial^*\breve{\sigma}_{n-1})$.)

Bulk/boundary symmetries in MBQS

In other words, the boundary global symmetry is promoted to the bulk(+boundary) global symmetry $X(\breve{z}^*_{d-n+1})|\psi_C\rangle = |\psi_C\rangle$, and it is gauged in the cluster state.



"Holographic interplay"

Summary and outlook

Summary/Outlook

- Graph states/cluster states is a class of stabilizer states that can be used for MBQC.
- The 2d cluster state on a regular lattice is a universal resource.
- *Open Question:* What is the precise characterization of an MBQC resource state? "Universal phase of quantum matter"?

- The cluster state entangler and measurements combined together offer a shortcut to deconfinement phases.
- The preparation of the toric code state was recently achieved with this method. We expect that more exciting results along this direction will come out in the near future.
- This can be potentially applied to quantum simulations as well.
- *Open Question:* How about for continuous gauge groups (e.g. $U(1)$) etc.? Cf. [Ashkenazi-Zohar (2021)]

Summary/Outlook

- I also explained an Measurement-Based Quantum Simulation scheme. Depending on properties of experimental devices, there can be some advantage over gate-based quantum simulations. E.g. run time.
- So far, this has been formulated for $\mathbb{Z}_N$ higher-form gauge theories in arbitrary dimensions, the Fradkin-Shenker model, and Kitaev's Majorana chain model.
- It is also possible to implement the imaginary-time evolution with post selections.

- *Open Question*: Can we formulate an MBQS for $U(1)$ lattice gauge theories and theories with Dirac/Weyl fermions?
- *Open Question*: Is the MBQS possible over the family of states within some SPT *phase* which includes the state $|\mathrm{gCS}\rangle$? (Similar to the notion of "universal phase of quantum matter")
- *Thoughts*: Relation to the overlap fermion formalism and its anomaly inflow?

Further readings

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- R. Raussendorf, C. Okay, D.-S. Wang, D. T. Stephen, and H. P. Nautrup, *A computationally universal phase of quantum matter, Phys. Rev. Lett. 122, 090501 (2019)*
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SPT in gCS

[Chen-Gu-Wen]

- A state has a **long-range entanglement** iff it is not short-range entangled.
- A state $|\Phi\rangle$ has a **short-range entanglement** iff there is (finite-depth) local unitary evolution such that $|\Phi\rangle = U|\Phi_{\text{prod}}\rangle$

SPT in gCS

[Chen-Gu-Wen]

- A state has a **nontrivial SPT order** if it is SRE and it is not a trivial SPT.
- A symmetric state $|\Phi\rangle$ has a **trivial SPT order** with respect to a symmetry G iff there is (finite-depth) symmetric local unitary evolution such that $|\Phi\rangle = U_{\text{sym}}|\Phi_{\text{prod}}\rangle$

SPT ordered state

Symmetric-SRE

Product state